

BROWN-ZAGIER RELATION FOR ASSOCIATORS

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1. INTRODUCTION

We have big heritage of equalities on hypergeometric functions, which can be used for showing many equalities for multiple zeta values. This method can be also applicable for showing relations between coefficients of associators using the theory of Φ -cohomology. A Φ -cohomology is equipped with two realizations B, dR and a comparison map described by the given associator Φ . Brown [B] used certain relation between multiple zeta values to show the injectivity of the homomorphism from Motivic Galois group to Grothendieck-Teichmuller group. This relation was proved by Zagier [Z], which we call Brown-Zagier relation. After his work, Li [L] gave another proof of Brown-Zagier relation using several functional equations of hypergeometric series.

In this paper, we show that Brown-Zagier relation holds also for the coefficients of any associators. In the paper [L], he proved Brown-Zagier relation using Dixon's theorem which is equivalent to Selberg integral formula. The Selberg integral formula arises from symmetric product construction, which does not exist in the category of moduli space. Even in this case, we can construct isomorphism between Φ -local systems using descent theory. The main theorem is stated as follows.

Theorem 1.1. *We use the notation for coefficients $\zeta_\Phi(n_1, \dots, n_m)$ of an associator Φ . Then we have*

$$\zeta_\phi(2^a, 3, 2^b) = 2 \sum_{r=1}^{a+b+1} (-1)^r c_{a,b}^r \zeta_\Phi(2r+1) \zeta_\Phi(2^{a+b-r+1}),$$

where

$$c_{a,b}^r = \binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1}$$

Since the generating series of motivic multiple zeta values satisfies the associator relation, we have the following corollary.

Corollary 1.2. *The same relation holds in the coordinate ring of mixed Tate motives.*

The above corollary gives an another proof of a result of Brown.

Notation 1.3. *The product of Gamma function $\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)$ is denoted by $\Gamma(a_2, a_2, \dots, a_n)$ for short.*

2. DIFFERENTIAL EQUATIONS AND GENERATING FUNCTIONS

In this section, we recall outline of classical theory of hypergeometric functions and Gauss-Manin connections.

2.1. Differential equation and iterated integral. Let N^* be a $\mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$ left module. The action of e_0 and e_1 on N^* is denoted by P_0 and P_1 . Let \mathcal{O}_U be the ring of analytic functions on an open set U in $\mathbf{P}^1 - \{0, 1, \infty\}$. We define a map $P : N^* \otimes \mathcal{O}_{an} \rightarrow N^* \otimes \Omega_{an}^1$

$$N^* \rightarrow N^* \otimes \langle \frac{dx}{x}, \frac{dx}{x-1} \rangle : v \mapsto P(x)v$$

where $P(x) = P_0 \frac{dx}{x} + P_1 \frac{dx}{x-1}$. There exists a unique local solution $\Phi_u(x)$ of the differential equation $d\Phi(x) = P(x)\Phi(x)$ for $\text{End}(N^*, N^*)$ -valued analytic functions such that $\Phi_u(u) = id_V$. It is denoted by $\exp(\int_u^x P)$. For any solution of the differential equation $dV = PV$ for $\text{End}(N^*, N^*)$ -valued functions, we have

$$V(x) = (I + \int_u^x P + \int_u^x PP + \dots)V(u) = \exp(\int_u^x P)V(u)$$

for $t \in \mathbf{R}, 0 < t_0 < \epsilon$. For a path γ from u to u' , $\exp(\int_u^{u'} P)$ depends only on the homotopy class of γ , which is denoted by $\rho(\gamma)$. Then ρ defines a left $\pi_1(\mathcal{M}_4)$ -module on N^* .

2.2. Differential equation of Gauss hypergeometric functions. In this section, we recall the differential equations satisfied by hypergeometric functions.

2.2.1. We define hypergeometric function by

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} x^n,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. The hypergeometric function has the following integral expression.

$$B(a, c-a)F(a, b; c; x) = \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-xt)^{-b} dt.$$

2.2.2. The differential is denoted by $D = \frac{\partial}{\partial x}$. We define a matrix $V_0 = (v_{ij})_{1 \leq i, j \leq 2}$, where

$$\begin{aligned} v_{11}^{(0)} &= \frac{\Gamma(a, c-a+1)}{\Gamma(c+1)} F(a, b; c+1; x), \\ v_{12}^{(0)} &= x^{-c} \frac{\Gamma(b-c, 1-b)}{\Gamma(1-c)} F(b-c, a-c; 1-c; x), \\ v_{22}^{(0)} &= \frac{1}{a} x D(v_{12}), \quad v_{21}^{(0)} = \frac{1}{a} x D(v_{11}) \end{aligned}$$

Let P be a matrix defined by

$$(2.1) \quad P = \frac{dx}{x}P_0 + \frac{dx}{x-1}P_1$$

where

$$P_0 = \begin{pmatrix} 0 & a \\ 0 & -c \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ -b & c-a-b \end{pmatrix}.$$

Then the matrix V_0 satisfies the differential equation

$$(2.2) \quad dV = PV.$$

Let V_1 be matrix defined by

$$\begin{aligned} v_{11}^{(1)} &= \frac{\Gamma(a, b-c)}{\Gamma(a+b-c)} F(a, b; a+b-c; 1-x), \\ v_{12}^{(1)} &= (1-x)^{c-a-b+1} \frac{\Gamma(c+1-a, 1-b)}{\Gamma(c+2-a-b)} F(c+1-a, c+1-b; 2+c-a-b; 1-x), \\ v_{22}^{(1)} &= \frac{1}{a} x D(v_{12}) \quad v_{21}^{(1)} = \frac{1}{a} x D(v_{11}), \end{aligned}$$

Then V_1 also satisfies the differential equation (2.2).

2.2.3. Connections and differential equations for coefficients. Let N^* be the vector space generated by ω_1^*, ω_2^* , N be its dual and ω_1, ω_2 be the dual basis of ω_1^*, ω_2^* . We define a linear maps $\nabla : N \rightarrow N \otimes \langle \frac{dx}{x}, \frac{dx}{x-1} \rangle$ and $\nabla^* : N^* \rightarrow N^* \otimes \langle \frac{dx}{x}, \frac{dx}{x-1} \rangle$ by

$$(2.3) \quad \begin{aligned} \nabla \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &= P \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ \nabla^* (\omega_1^* \quad \omega_2^*) &= -(\omega_1^* \quad \omega_2^*) P \end{aligned}$$

The map ∇^* can be extended to a connection on the $N^* \otimes \mathbf{C}[x, \frac{1}{x}, \frac{1}{x-1}]$, which is also denoted by ∇^* . We use the following identification

$$f_1(x)\omega_1^* + f_2(x)\omega_2^* = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

Then we have

$$\nabla \begin{pmatrix} f_1(x)dx \\ f_2(x)dx \end{pmatrix} = d \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} - P \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

on $N^* \otimes \mathbf{C}[x, \frac{1}{x}, \frac{1}{x-1}]$. Let γ be an N^* -valued analytic function. Using the pairing $\langle \cdot, \cdot \rangle$, γ is written as

$$\gamma = \langle \gamma, \omega_1 \rangle \omega_1^* + \langle \gamma, \omega_2 \rangle \omega_2^* = \begin{pmatrix} \langle \gamma, \omega_1 \rangle \\ \langle \gamma, \omega_2 \rangle \end{pmatrix},$$

Therefore γ is a horizontal section for ∇ if and only if

$$d \begin{pmatrix} \langle \gamma, \omega_1 \rangle \\ \langle \gamma, \omega_2 \rangle \end{pmatrix} = P \begin{pmatrix} \langle \gamma, \omega_1 \rangle \\ \langle \gamma, \omega_2 \rangle \end{pmatrix}$$

2.2.4. *Hypergeometric function and its integral expression.* By the integral expression of hypergeometric functions, the matrix elements of V_0 is written as

$$(2.4) \quad \begin{aligned} v_{11}^{(0)} &= \int_{\gamma_1} \omega_1, v_{12}^{(0)} = \int_{\gamma_2} \omega_1, v_{22}^{(0)} = \int_{\gamma_1} \omega_2, v_{22}^{(0)} = \int_{\gamma_2} \omega_2, \\ v_{11}^{(1)} &= \int_{\gamma_1^\#} \omega_1, v_{12}^{(1)} = \int_{\gamma_2^\#} \omega_1, v_{22}^{(1)} = \int_{\gamma_1^\#} \omega_2, v_{22}^{(1)} = \int_{\gamma_2^\#} \omega_2, \end{aligned}$$

where ω_1, ω_2 are the relative twisted de Rham cohomology classes defined by

$$(2.5) \quad \omega_1 = \left[\frac{dt}{t} \right], \quad \omega_2 = \left[\frac{bxdt}{a(1-xt)} \right],$$

and γ_1, γ_2 be twisted cycle defined by

$$(2.6) \quad \begin{aligned} \gamma_1 &= [t^a(1-t)^{c-a}(1-xt)^{-b}]_{[0,1]}, \\ \gamma_2 &= [t^a(t-1)^{c-a}(xt-1)^{-b}]_{[\frac{1}{x}, \infty]} \\ \gamma_1^\# &= [(-t)^a(1-t)^{c-a}(1-xt)^{-b}]_{[-\infty,0]} \\ \gamma_2^\# &= [t^a(t-1)^{c-a}(1-xt)^{-b}]_{[1, \frac{1}{x}]} \end{aligned}$$

We have the following equality of cycles.

$$(2.7) \quad \begin{aligned} \mathbf{s}(c)\gamma_1^\# &= \mathbf{s}(c-a)\gamma_1 + \mathbf{s}(b)\gamma_2, \\ \mathbf{s}(c-a-b)\gamma_1 &= \mathbf{s}(c-b)\gamma_1^\# + \mathbf{s}(b)\gamma_2^\#. \end{aligned}$$

where $\mathbf{s}(z) = \frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}$. Let N be the vector space generated by ω_1, ω_2 . Then the Gauss-Manin connection is given by the map (2.3). Under the comparison map, γ_1, γ_2 defines a horizontal N^* -valued analytic map on $(0, 1)$. By the expression 2.4, we have

$$\begin{aligned} \lim_{\epsilon \rightarrow +0} \gamma_1(\epsilon) &= B(a, c-a+1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \lim_{\epsilon \rightarrow +0} (\epsilon^c \gamma_2)(\epsilon) &= B(b-c, 1-b) \begin{pmatrix} 1 \\ -\frac{c}{a} \end{pmatrix}. \end{aligned}$$

2.3. Gauss-Manin connection and horizontal section on the daul.

2.3.1. *Dual differential equations.* We construct solutions of the dual differential equation around 1. We define a matrix $W_1 = (w_{ij})_{1 \leq i,j \leq 2}$ by

$$\begin{aligned} w_{11} &= \frac{\Gamma(-a, c-b+1)}{\Gamma(c-a-b+1)} F(-a, -b; c-a-b+1; 1-x), \\ w_{21} &= \frac{\Gamma(a-c, b+1)}{\Gamma(a+b-c+1)} (1-x)^{-c+a+b} F(a-c, b-c; 1-c+a+b; 1-x) \\ w_{12} &= -\frac{1}{b} (1-x) D(w_{11}), \quad w_{22} = -\frac{1}{b} (1-x) D(w_{21}). \end{aligned}$$

The matrix elements of W_1 is written as

$$v_{11} = \int_{\gamma_1^*} \omega_1^*, v_{12} = \int_{\gamma_2^*} \omega_1^*, v_{21} = \int_{\gamma_1^*} \omega_2^*, v_{22} = \int_{\gamma_2^*} \omega_2^*,$$

where ω_1^*, ω_2^* are

$$(2.8) \quad \omega_1^* = \left[\frac{dt}{t} \right], \quad \omega_2^* = \left[\frac{(x-1)dt}{(1-xt)} \right],$$

and γ_1^*, γ_2^* are

$$\begin{aligned} \gamma_1^* &= [(-t)^{-a}(1-t)^{-c+a}(1-xt)^b]_{[-\infty,0]} \\ \gamma_2^* &= [t^{-a}(t-1)^{-c+a}(1-xt)^b]_{[1,\frac{1}{x}]} \end{aligned}$$

Then the matrix W_1 satisfies the differential equation $dW_1 = -W_1 P$. Therefore we have

$$W_1 = W_1(t_1) \exp\left(\int_x^{t_1} P\right).$$

2.3.2. Duality and exponential map around 1. Let V_1 and W_1 be matrix defined in §2.2, §2.3.1. Since

$$\frac{\partial}{\partial x}(W_1 V_1) = \frac{\partial W_1}{\partial x} V_1 + W_1 \frac{\partial V_1}{\partial x} = -W_1 P V_1 + W_1 P V_1 = 0$$

the matrix $W_1 V_1$ does not depends on x . By considering the limit for $x \rightarrow 0$, we have

$$(2.9) \quad W_1 V_1 = \begin{pmatrix} \frac{s(a+b-c)}{as(-a)s(b-c)} & 0 \\ 0 & \frac{s(c-a-b)}{as(a-c)s(b)} \end{pmatrix} = D_1$$

and as a consequence, we have

$$V_1(y) D_1^{-1} W_1(x) = \exp \int_x^y P.$$

2.4. Generating function of multiple zeta values $\zeta(2, \dots, 3, \dots, 2)$. We specialize to the case $c = 0, a = -b$. Then the matrix P_0, P_1 of (2.1) becomes

$$(2.10) \quad P_0 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

Using the limit computation of the last subsection, we have

$$(s(a) \ 0) V_0(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \ 0) \exp\left(\int_0^x P\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F(a, -a; 1; x)$$

Proposition 2.1. *Let P_0 and P_1 be matrices defined in (2.10).*

(1) *For $I = (i_1, \dots, i_n) \in \{0, 1\}^n$, we define $E_I = P_{i_1} \cdots P_{i_n}$. Then we have*

$$(1, 0) P_I \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} a^{2n} & \text{if } I = (10)^n \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let $\varphi(e_0, e_1)$ be an element $\mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$ given by

$$(2.11) \quad \varphi(e_0, e_1) = \sum_{n \geq 0} \sum_{i_1 \in \{0,1\}, \dots, i_n \in \{0,1\}} c_{i_1, \dots, i_n} e_{i_1} \dots e_{i_n}$$

where $c_{i_1, \dots, i_n} \in \mathbf{C}$. Then we have

$$(2.12) \quad (1, 0)\varphi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + \sum_{n > 0} c_{(01)^n} a^{2n}.$$

Proof. This is an easy consequence of the equalities $P_0^2 = P_1^2 = 0$, and

$$P_0 P_1 = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 P_0 = \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix}.$$

□

Since $V(x)$ is expressed by using iterated integral, we have

$$\begin{aligned} F(a, -a; 1; x) &= (1 \ 0) \sum_{n=0}^{\infty} \int_0^x (P_0 \frac{du}{u} + P_1 \frac{du}{u-1})^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \int_0^x (\frac{du}{u} \frac{du}{u-1})^n a^{2n} \end{aligned}$$

by Proposition 2.1. By setting $x = 1$, we have

$$\sum_{n=0}^{\infty} \int_0^1 (\frac{du}{u} \frac{du}{u-1})^n a^{2n} = F(a, -a; 1; 1) = \frac{\sin(\pi a)}{\pi a}$$

by the equality (??). Similarly, we have

$$B(-a, a+1)^{-1} (1 \ 0) W_1(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \ 0) \exp(\int_x^1 P) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F(-a, a, 1, 1-x),$$

and it is equal to

$$\sum_{m=0}^{\infty} \int_x^1 (\frac{du}{u} \frac{du}{u-1})^m a^{2m}$$

We have the following proposition.

Proposition 2.2. *We set*

$$(2.13) \quad \phi(a, b) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^1 (\frac{du}{u} \frac{du}{u-1})^m \frac{du}{u} (\frac{du}{u} \frac{du}{u-1})^n a^{2n} b^{2m}$$

Then we have

$$\phi(a, b) = \int_0^1 F(-b, b, 1, 1-w) (F(a, -a, 1, w) - 1) \frac{dw}{w}$$

Proof. By the definition of iterated integral, we have

$$\begin{aligned} & \int_0^1 \left(\frac{du}{u} \frac{du}{u-1} \right)^m \frac{du}{u} \left(\frac{du}{u} \frac{du}{u-1} \right)^n \\ &= \int_0^1 \left[\int_x^1 \left(\frac{du}{u} \frac{du}{u-1} \right)^m \right] \frac{dx}{x} \left[\int_1^x \left(\frac{dv}{v} \frac{dv}{v-1} \right)^n \right] dx. \end{aligned}$$

By taking the generating function on m and n , we get the proposition. \square

Remark 2.3. Zagier showed that $\phi(a, b)$ is also equal to

$$(2.14) \quad \frac{\sin(\pi b)}{\pi b} \frac{d}{dz} |_{z=0} {}_3F_2(a, -a, z; 1+b, 1-b; 1).$$

In §6, we show that associator versions of the formal power series (2.13) and (2.14) coincides.

3. ASSOCIATOR AND HOPF ALGEBROID

3.1. Fundamental algebroid of moduli spaces. We recall the structure of Hopf algebroids $\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}$ of the moduli space $\mathcal{M}_n = \mathcal{M}_{0,n}$ of n -punctured genus zero curves in this subsection.

Definition 3.1. We define the set of tangential points T_n of n points in genus zero curve as the set of planer trivalent tree with n terminals. For example

$$T_4 = \{\overline{01}, \overline{10}, \overline{0\infty}, \overline{\infty0}, \overline{1\infty}, \overline{\infty1}\}$$

Thus $\#T_4 = 3 \times 2$, $\#T_5 = 15 \times 4$, etc.

Then we can define the pro-nilpotent algebroid $\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}$ over the set T_n as follows.

Definition 3.2. For two points $a, b \in T_n$, the bifiber of the algebroid $\mathcal{A}_{n,dR} = \{\mathcal{A}_{n,dR,ab}\}_{ab}$ is defined as the following generators and relations.

- (1) (Generators) t_{ij} with $1 \leq i < j \leq n$. We use the notation $t_{ji} = t_{ij}$ for $i < j$.
- (2) (Relations)
 - (a) $[t_{ij}, t_{kl}] = 0$
 - (b) $[t_{ij}, t_{ik} + t_{kj}] = 0$
 - (c) $\sum_{j \neq i} t_{ij} = 0$

Then $\mathcal{A}_{n,dR}$ is the completed de Rham fundamental group algebra of \mathcal{M}_n and has a standard coproduct $\Delta(t_{ij}) = t_{ij} \otimes 1 + 1 \otimes t_{ij}$.

Definition 3.3. (1) Two tangential base points $a, b \in T_n$ are adjacent if it can be transformed by elementary change $H \leftrightarrow I$.
(2) Two tangential base points $a, b \in T_n$ are neighbours if it can be transformed by twisting with respect to a edge.
(3) $\mathcal{A}_{n,B} = \{\mathcal{A}_{n,B,ab}\}_{ab}$ is a pro-nilpotent algebroid generated by two type of generators:

- (a) path p_{ab} connecting two adjacent tangential base points.
- (b) small circle c_{ab} connecting two neighbours.

(c) Relations on $\mathcal{A}_{n,B}$ are generated by 2-cycle relations, 3-cycle relations, 5-cycle relations.

Then the $\mathcal{A}_{n,B}$ is the completed groupoid algebra of \mathcal{M}_n .

Definition 3.4. (Category \mathcal{C}) We define the abelian category \mathcal{C} as follows. An object V of \mathcal{C} is a triple (V_{dR}, V_B, c_V) consisting of

- (1) \mathbf{Q} -vector space V_{dR} ,
- (2) \mathbf{Q} -vector space V_B , and
- (3) an isomorphism $V_B \otimes \mathbf{C} \simeq V_{dR} \otimes \mathbf{C}$

Sometimes one consider profinite version. In this case, $\otimes \mathbf{C}$ means the completed tensor product. Morphism form $f : V \rightarrow W$ is a pair of morphisms $f_{dR} : V_{dR} \rightarrow W_{dR}$ and $f_B : V_B \rightarrow W_B$ compatible with the comparison maps. The category \mathcal{C} becomes a tensor category by tensoring each dR and B components

Definition 3.5. We define the category M^{inf} be the category whose objects are \mathcal{M}_n and morphisms are generated by infinitesimal inclusions.

Definition 3.6. We can define two functors $\mathcal{A}_{dR}, \mathcal{A}_B : M^{inf} \rightarrow \text{Hopf}_{\mathbf{Q}}$ from M^{inf} to the category of Hopf algebroids by attaching de Rham fundamental groups and Betti fundamental groups.

3.2. Choice of coordinate. Let C be a genus zero curve and $P = (C, p_1, \dots, p_n)$ ($p_i \in C$) an element in \mathcal{M}_n . We choose a coordinate t of C such that $t(p_{n-2}) = 0, t(p_{n-1}) = 0, t(p_n) = 0$. Using the coordinate t , \mathcal{M}_n is identified with an open set of \mathbf{A}^{n-3} defined by

$$\{(x_1, \dots, x_{n-3}) \mid x_i \neq x_j \text{ for } i \neq j, x_i \neq 0, 1 \text{ for all } i\}$$

by setting $x_k = t(p_k)$. This coordinate is called the distinguished coordinate. By taking the distinguished coordinate of \mathcal{M}_4 , the underlying curve is identified with $\mathbf{P}^1 - \{0, 1, \infty\}$.

Definition 3.7 (admissible function, admissible differential form). (1) Let

$S = (i, j, k, l)$ be a ordered subset of distinct elements in $[1, n]$. For an element $P = (C, p_1, \dots, p_n)$ be an element of \mathcal{M}_n . There is a unique coordinate t of C such that $t(p_i) = 0, t(p_j) = 1, t(p_k) = \infty$. The value $t(p_l)$ at p_l gives rise to an algebraic function on \mathcal{M}_n , which is denoted by φ_S . The set of admissible functions is denoted by $\text{Ad}(\mathcal{M}_n)$.

(2) Let x_1, \dots, x_{n-3} be the distinguished coordinate. An element in the linear span of $\frac{dx_i}{x_i}, \frac{dx_i}{x_i-1}, \frac{d(x_i-x_j)}{x_i-x_j}$ is called an admissible differential form.

Remark 3.8. (1) $\varphi \in \text{Ad}(\mathcal{M}_n)$ defines a morphism $\mathcal{M}_n \rightarrow \mathcal{M}_4$. and a morphism of algebroids $\mathcal{A}_n \rightarrow \mathcal{A}_4$.

(2) If $S \cap \{n-2, n-1, n\} = \emptyset$, using the distinguished coordinates of \mathcal{M}_n , we have

$$\varphi_S(P) = \frac{(x_l - x_i)(x_j - x_k)}{(x_l - x_k)(x_j - x_i)}.$$

Therefore φ_S is invariant under substitutions $i \leftrightarrow l, j \leftrightarrow k$ and $i \leftrightarrow j, k \leftrightarrow l$.

(3) The following functions are admissible functions.

$$\frac{x_i}{x_j} = \frac{(x_i - 0)(x_j - \infty)}{(x_j - 0)(x_i - \infty)}, \quad 1 - \frac{x_i}{x_j} = \frac{(x_j - x_i)(\infty - 0)}{(x_j - 0)(\infty - x_j)}.$$

$$1 - x_i = \frac{(1 - x_i)(\infty - 0)}{(1 - 0)(\infty - x_i)}$$

Proposition 3.9. The set of functorial isomorphisms from $\mathcal{A}_B \otimes \mathbf{C}$ to $\mathcal{A}_{dR} \otimes \mathbf{C}$ sending small half circle $\log(c_{ij})$ to $\pi i t_{ij}$ is identified with the set of assoicators. The one to one correspondence is given by

$$\mathcal{A}_{4,B,\overline{01},\overline{10}} \ni [0,1] \mapsto \Phi \in \mathcal{A}_{dR,4} = \mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$$

Here e_0 and e_1 are the dual basis of $\omega_0 = \frac{dx}{x}$ and $\omega_1 = \frac{dx}{x-1}$, respectively.

By the above proposition, we have an isomorphism of Hopf algebra

$$c_{\Phi,n} : \mathcal{A}_{n,B} \otimes \mathbf{C} \xrightarrow{\cong} \mathcal{A}_{n,dR} \otimes \mathbf{C}.$$

associated to a given assoicator Φ . This isomorphism gives an object $\mathcal{A}_n^\Phi = (\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}, c_{\Phi,n})$. The isomorphism $c_{\Phi,n}$ is called the Φ -comparison map.

Proposition 3.10. (1) Let $3 \leq m < n$ be integers and f morphsim defined by

$$f : \mathcal{M}_n \rightarrow \mathcal{M}_m : (x_1, \dots, x_{n-3}) \rightarrow (x_1, \dots, x_{m-3})$$

Then for $\star = dR, B$, the induced maps of algebroids

$$\mathcal{A}_{n,\star} \rightarrow \mathcal{A}_{m,\star}$$

are compatible with the Φ -comparison maps.

(2) Let $3 \leq m, n$ be integers. Then a morphsim

$$f : \mathcal{M}_{n+m-3} \rightarrow \mathcal{M}_n \times \mathcal{M}_m$$

$$(x_1, \dots, x_{n-3}, y_1, \dots, y_{m-3}) \mapsto (x_1, \dots, x_{n-3}) \times (y_1, \dots, y_{m-3})$$

induces a morphism of algebroids

$$f : \mathcal{A}_{n+m-3} \rightarrow \mathcal{A}_n \otimes \mathcal{A}_m$$

in \mathcal{C} .

(3) Let $3 \leq m < n_1, n_2$ be integers. Then the natural morphsim

$$f : \mathcal{M}_{n_1} \times_{\mathcal{M}_m} \mathcal{M}_{n_2} \rightarrow \mathcal{M}_{n_1} \times \mathcal{M}_{n_2}$$

induces a morphism of algebroids in \mathcal{C} .

The coefficient $c_{\Phi,I}$ of $e_{i_1} e_{i_2} \dots e_{i_k}$ in $c_{\Phi,4}([0,1])$ is written as $\int_{[0,1]}^\Phi \omega_{i_1} \dots \omega_{i_k}$. We define Φ -multiple zeta value similarly. A Φ -multiple zeta value is written as

$$\zeta_\Phi(m_1, \dots, m_k) = \int_{[0,1]}^\Phi \omega_0^{m_k-1} \omega_1 \dots \omega_0^{m_1-1} \omega_1$$

It is a coefficint of the associator Φ .

3.3. \mathcal{A} -module. Let T be a set and \mathcal{A} a Hopf algebroid object in \mathcal{C} over T . We define the notion of \mathcal{A} -module.

Definition 3.11. Let $M = (M_a)_{a \in T} = (M_{dR,a}, M_{B,a}, c_{M,a})_{a \in T}$ be an object in \mathcal{C} indexed by $a \in T$. M is called an \mathcal{A} -module if it is equipped with an action of \mathcal{A} in \mathcal{C}

$$\mu_M : \mathcal{A} \otimes M \rightarrow M$$

which is associative and unitary. Here action of algebroid is given by a morphism

$$\mathcal{A}_{ab} \otimes M_a \rightarrow M_b.$$

in \mathcal{C} .

Remark 3.12. Let M, N be \mathcal{A} module. Then using coproduct structure of \mathcal{A} , $M \otimes N$ is equipped with \mathcal{A} module.

Example 3.13. (1) Let $4 \leq m < n$ and $f : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be the map defined by $(x_1, \dots, x_{n-3}) \mapsto (x_1, \dots, x_{m-3})$. Then we have an algebroid homomorphism $f : \mathcal{A}_n^\Phi \rightarrow \mathcal{A}_m^\Phi$. Therefore for a fixed $p \in T_m$, by setting $M_a = \mathcal{A}_{m,p,f(a)}^\Phi$ we have an \mathcal{A}_n^Φ -module. It is called a pull back of the map f .
(2) By taking an abelianization $\mathcal{A}_n^{\Phi,ab}$ of \mathcal{A}_n^Φ , we have a homomorphism of Hopf algebroids

$$\mathcal{A}_n^\Phi \rightarrow \mathcal{A}_n^{\Phi,ab}.$$

By choosing a base point $p \in T_n$, we have have an \mathcal{A}_n^Φ -module $\mathcal{A}_{n,p*}^{\Phi,ab}$. In particular, by using the distinguished coordinate x , \mathcal{A}_4 module $x^\alpha \mathbf{Q}[[a]]$ is defined by taking the base point as $\overline{01}$,

(3) Let φ be an admissible function on \mathcal{M}_n and α formal parameter. The morphism $\mathcal{M}_n \rightarrow \mathcal{M}_4$ induced by φ is also denoted by φ and x be the distinguished coordinate of \mathcal{A}_4 . We define $\mathcal{A}_n[[\alpha]]$ -module

$$\varphi^\alpha \mathbf{Q}[[\alpha]]$$

by the pull back $\varphi^*(x^\alpha \mathbf{Q}[[a]])$ of $x^\alpha \mathbf{Q}[[a]]$. We define

$$\left(\prod_{i=1}^m \varphi_i^{\alpha_i} \right) \mathbf{Q}[[\alpha_1, \dots, \alpha_m]] = \varphi_1^{\alpha_1} \mathbf{Q}[[\alpha_1]] \widehat{\otimes} \cdots \widehat{\otimes} \varphi_m^{\alpha_m} \mathbf{Q}[[\alpha_m]]$$

Proposition 3.14. Let φ_i , ($i = 1, \dots, m$), ψ_j , ($j = 1, \dots, l$) be admissible functions on \mathcal{M}_n and $a_{ij} \in \mathbf{Z}$. We assume that $\psi_j = \prod_{i=1}^m \varphi_i^{a_{ij}}$. We set

$$L_j = \sum_i^m a_{ij} \alpha_i$$

for $j = 1, \dots, l$. Then

$$\left(\prod_{i=1}^m \varphi_i^{\alpha_i} \right) \mathbf{Q}[[\alpha_i]] = \left(\prod_{j=1}^l \psi_j^{L_j} \right) \mathbf{Q}[[\alpha_i]].$$

as \mathcal{A}_n^Φ module.

Let \mathcal{A} be an algebraoid in \mathcal{C} and M be an \mathcal{A} -module. We define the dual M^* of M using antipodal.

Proposition 3.15 (Descent theory for \mathcal{A}_4^Φ -module.). *Let M be an \mathcal{A}_4^Φ -module. Assume that M_{dR} is constant, i.e. the map*

$$M_{dR} \xrightarrow{e_0, e_1} M_{dR} \oplus M_{dR}$$

*is the zero map. Then M is the pull back of an object N in \mathcal{C} such that $M = \pi^*N$, where $\pi : \mathcal{A}_4^\Phi \rightarrow \mathbf{Q}$ is the augmentation map.*

Proof. Let $I = \ker(A_4^\Phi \rightarrow \mathbf{Q})$ be the augmentation ideal. Then we have the following commutative diagram whose vertical arrows come from comparison maps and are isomorphisms.

$$\begin{array}{ccc} I_{B,ab} \otimes M_{B,a} \otimes \mathbf{C} & \xrightarrow{\alpha} & M_{B,b} \otimes \mathbf{C} \\ \downarrow & & \downarrow \\ I_{dR,ab} \otimes M_{dR,a} \otimes \mathbf{C} & \xrightarrow{\beta} & M_{dR,b} \otimes \mathbf{C} \end{array}$$

Since β is the zero map, α is the zero map. Therefore M is induced from an object in \mathcal{C} . \square

3.4. Comparison map and actions.

3.4.1. *de Rham framing.* Let M be an $\mathcal{A}_4^\Phi[[\alpha_i]]$ -module. and $c_M : M_B \rightarrow M_{dR}$ be the comparison map of M .

Definition 3.16. (1) *Let $y \in T_4$. A de Rham framing of M is a pair of homomorphisms $\alpha : \mathbf{Q}[[\alpha_i]] \rightarrow M_{B,y}$ and $\beta : M_{dR} \rightarrow \mathbf{Q}[[\alpha_i]]$ of $\mathbf{Q}[[\alpha_i]]$ -modules.*
(2) *Let $f = (\alpha, \beta)$ be a framing of M at y , and γ be an element in $\mathcal{A}_{4,B,yz}^\Phi$ the value $f(\gamma)$ of f at γ is defined by*

$$\beta \circ c_M \circ \gamma \circ \alpha \in \mathbf{Q}[[\alpha_i]].$$

Let $f = (\alpha, \beta)$ be a framing of M at $\overline{01}$. The dR -part M_{dR} of M is a $\mathcal{A}_{4,dR}^\Phi \simeq \mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$ module. Let E_0, E_1 be actions of e_0 and e_1 on M_{dR} . The action of $\varphi = \varphi(e_0, e_1) \in \mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$ on M_{dR} is denoted by $\varphi(E_0, E_1)$. Since the actions of $\mathcal{A}_{4,B}^\Phi$ and $\mathcal{A}_{4,dR}^\Phi$ on M_B and M_{dR} are compatible via the comparison map, using the associator Φ , we have

$$f([0, 1]) = \beta c_M[0, 1] \alpha = \beta c_{\mathcal{A}_4^\Phi}([0, 1]) c_M \alpha = \beta \Phi(E_0, E_1) c_M \alpha \in \mathbf{Q}[[\alpha_i]].$$

3.4.2. *Example 1.* We consider a module $M_{dR} = \mathbf{Q}[[a]]^{\oplus 2}$. Let P_0, P_1 be endomorphisms defined as (2.10). Therefore it defines a $\mathcal{A}_{4,dR}$ module structure on M_{dR} . The action of φ of (2.11) is given by (2.12).

3.4.3. *Example 2.* We consider a module $M_{dR} = \mathbf{Q}[[a, b]]^{\oplus 2}$. Let P_0, P_1 be endomorphisms defined by

$$(3.1) \quad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ -b & c-b \end{pmatrix}.$$

of M_{dR} . Then it defines a $\mathcal{A}_{4,dR}$ module structure on M_{dR} . For $I = (i_1, \dots, i_n) \in \{0, 1\}^n$, we have

$$(0, 1)P_I \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} 0 & \text{if } i_n = 0 \\ (-b)(-c)^p(c-b)^q & \text{if } i_n = 1 \text{ where } p = \#\{i_k = 0\} - 1, q = \#\{i_k = 1\}, \end{cases}$$

Proposition 3.17. *Let*

$$\varphi(e_0, e_1) = 1 + \varphi_0(e_0, e_1)e_0 + \varphi_1(e_0, e_1)e_1$$

be an element $\mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$. Then we have

$$(0, 1)\varphi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-b)\varphi_1^{ab}(-c, c-b) \in \mathbf{C}[[b, c]].$$

where $\varphi_0^{ab}(-b, c-b)$ is the image under the abelianization map $\mathbf{C}\langle\langle e_0, e_1 \rangle\rangle \rightarrow \mathbf{C}[[b, c]]$.

4. HIGHER DIRECT IMAGES FOR \mathcal{A}^Φ -MODULES

In this section, we define relative cohomologies and study their properties.

4.1. Relative cohomology. In this section, \mathcal{A}_n is \mathcal{A}_n^Φ , $\mathcal{A}_{n,dR}^\Phi$ or $\mathcal{A}_{n,B}^\Phi$. Let $4 \leq m < n$ and M be an \mathcal{A}_n -module. Let $f : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be a map defined by

$$(x_1, \dots, x_{n-3}) \mapsto (x_1, \dots, x_{m-3})$$

and $f : \mathcal{A}_n \rightarrow \mathcal{A}_m$ be the induced morphism of algebroid objects in \mathcal{C} . We define a complex $F(\mathcal{A}_n)$ by

$$\begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{A}_n \otimes \mathcal{A}_n \otimes \mathcal{A}_n & \rightarrow & \mathcal{A}_n \otimes \mathcal{A}_n & \rightarrow & 0 \\ & & x \otimes y \otimes z & \mapsto & xy \otimes z - x \otimes yz & & \end{array}$$

Then the map $\mathcal{A}_n \otimes \mathcal{A}_n \rightarrow \mathcal{A}_n$ defined by $x \otimes y \mapsto xy$ defines a free $(\mathcal{A}_n \otimes \mathcal{A}_n^0)$ resolution $F(\mathcal{A}_n) \rightarrow \mathcal{A}_n$. Therefore $F(\mathcal{A}_n) \otimes_{\mathcal{A}_n} \mathcal{A}_m$ is a free \mathcal{A}_n -resolution of \mathcal{A}_m . We note the relation

$$Hom_{\pi_1(\mathcal{M}_n)}(\mathcal{A}_{m,B}, M_B) = M_B^N,$$

whrer $N = \ker(\pi_1(\mathcal{M}_n) \rightarrow \pi_1(\mathcal{M}_m))$. Motivated by the above relation, we define $\mathbf{R}f_* M$ by $Hom_{\mathcal{A}_n}(F(\mathcal{A}_n) \otimes_{\mathcal{A}_n} \mathcal{A}_m, \mathcal{M})$. More concretely, we have

$$(4.1) \quad \begin{aligned} \mathbf{R}f_* M : Hom(\mathcal{A}_m, \mathcal{M}) &\xrightarrow{d^0} Hom(\mathcal{A}_n \otimes \mathcal{A}_m, \mathcal{M}) \\ &\xrightarrow{d^1} Hom(\mathcal{A}_n \otimes \mathcal{A}_n \otimes \mathcal{A}_m, \mathcal{M}) \xrightarrow{d^2} \dots \end{aligned}$$

Here d^0 is given by $d^0(\varphi)(x \otimes y) = x\varphi(y) - \varphi(f(x)y)$. The right \mathcal{A}_m action on $F(\mathcal{A}_n) \otimes_{\mathcal{A}_n} \mathcal{A}_m$ induces a left \mathcal{A}_m -module structure on $\mathbf{R}f_* M$. As a

consequence, we have a left \mathcal{A}_m module $\mathbf{R}^i f_* M = H^i(\mathbf{R} f_* M)$. If $M = (M_{dR}, M_B, c_M)$ is an \mathcal{A}_n^Φ -module, then

$$(\mathbf{R}^i f_* M)_{dR} = \mathbf{R}^i f_*(M_{dR}), \quad (\mathbf{R}^i f_* M)_B = \mathbf{R}^i f_*(M_B)$$

If $f : \mathcal{M}_n \rightarrow pt = \mathcal{M}_3$, $\mathbf{R}^i f_* M$ is denoted by $H_\Phi^i(\mathcal{M}_n, M)$

4.2. Hochschild-Serre-Leray spectral sequence. Let $4 \leq l < m < n$ be natural numbers and $\mathcal{M}_n \xrightarrow{g} \mathcal{M}_m \xrightarrow{f} \mathcal{M}_l$ be a map defined by $(x_1, \dots, x_{n-3}) \mapsto (x_1, \dots, x_{m-3}) \mapsto (x_1, \dots, x_{l-3})$

Proposition 4.1. *The homomorphism*

$$F(\mathcal{A}_n) \otimes_{\mathcal{A}_n} (F(\mathcal{A}_m) \otimes_{\mathcal{A}_m} \mathcal{A}_l) \rightarrow F(\mathcal{A}_n) \otimes_{\mathcal{A}_n} \mathcal{A}_l$$

induces a quasi-isomorphism

$$\mathbf{R}(fg)_* M \xrightarrow{\sim} \mathbf{R} f_*(\mathbf{R} g_* M).$$

4.3. Fundamental algebroid of fibers and higher direct image.

4.3.1. Fibers of higher direct images. We give a method to compute the higher direct image for $f : \mathcal{M}_n \rightarrow \mathcal{M}_m$ for dR and B . Let $f : T_n \rightarrow T_m$ be the corresponding map for infinitesimal points, and y an element of T_m . We set $T_{n,m}(y) = f^{-1}(y)$.

Definition 4.2. Let $\mathcal{A}_{n,m,B,y}$ (resp. $\mathcal{A}_{n,m,dR,y}$) be the subalgebroid of $\mathcal{A}_{n,B}$, (resp. $\mathcal{A}_{n,dR}$) generated by the images of $\mathcal{A}_{4,B}$ induced by infinitesimal inclusions of $\mathcal{M}_4 \rightarrow \mathcal{M}_n$ contained in the fiber of y . Then the image of $\mathcal{A}_{n,m,B,y} \otimes \mathbf{C}$ is equal to $\mathcal{A}_{n,m,dR,y} \otimes \mathbf{C}$. Therefore $\mathcal{A}_{n,m,B,y}$ and $\mathcal{A}_{n,m,dR,y}$ defines a Hopf algebroid object in \mathcal{C} on $T_{n,m}(y)$, which is denoted by $\mathcal{A}_{n,m,y}$. For $x \in T_{n,m}(y)$, $\mathcal{A}_{n,m,y,x}$ is denoted by $\mathcal{A}_{n,m,x}$.

Remark 4.3. The B -part $\mathcal{A}_{n,m,B}$ can be interpreted as follows. Let $N_{n,m}$ be the kernel of $\pi_1^B(\mathcal{M}_n) \rightarrow \pi_1^B(\mathcal{M}_m)$. Then $N_{n,m}$ becomes a fibered groupoid over the map $T_n \rightarrow T_m$. We can easily see that $\mathcal{A}_{n,m}$ is the nilpotent completion of $N_{n,m}$.

Proposition 4.4. We choose $x \in T_n, y \in T_m$ such that $f(x) = y$. We have the following exact sequence:

$$0 \leftarrow \mathcal{A}_{m,y} \leftarrow \mathcal{A}_{n,x} \xleftarrow{d_0} \mathcal{A}_{n,x} \otimes \mathcal{A}_{n,m,x} \xleftarrow{d_1} \mathcal{A}_{n,x} \otimes \mathcal{A}_{n,m,x} \otimes \mathcal{A}_{n,m,x} \leftarrow \cdots$$

Here $d_0(x \otimes y) = xy - x\epsilon(y)$, $d_1(x \otimes y \otimes z) = xy \otimes z - x \otimes yz + x \otimes y\epsilon(z)$, \dots , where $\epsilon : \mathcal{A}_{n,m,x} \rightarrow \mathbf{Q}$ is the augmentation. This becomes a free $\mathcal{A}_{n,x}$ resolution of $\mathcal{A}_{m,y}$.

Proof. We reduce the proposition to the B -part. Let $f : G \rightarrow H$ be a surjective homomorphism of group and N be the kernel of f . We prove that the sequence

$$(4.2) \quad 0 \leftarrow \mathbf{Q}[H] \leftarrow \mathbf{Q}[G] \xleftarrow{d_0} \mathbf{Q}[G \times N] \xleftarrow{d_1} \mathbf{Q}[G \times N^2] \leftarrow \cdots$$

is exact. We choose a set theoretic section $s : H \rightarrow G$. Then

$$\begin{aligned}\theta_0 &: \mathbf{Q}[H] \rightarrow \mathbf{Q}[G] : h \mapsto s(h) \\ \theta_1 &: \mathbf{Q}[G] \rightarrow \mathbf{Q}[G \times N] : g \mapsto g \otimes g^{-1}s(g) \\ \theta_2 &: \mathbf{Q}[G \times N] \rightarrow \mathbf{Q}[G \times N^2] : g \otimes n \mapsto g \otimes n \otimes n^{-1}g^{-1}s(ng) \\ &\dots\end{aligned}$$

gives a null homotopy. Therefore the sequence (4.2) is an exact sequence. By taking a nilpotent completion, we have the proposition for the B -part. \square

Corollary 4.5. *The complex $\mathbf{R}f_*M_y$ is quasi-isomorphic to the complex $\text{Hom}_{\mathcal{A}_{n,m,x}}(F(\mathcal{A}_{n,m,x}) \otimes_{\mathcal{A}_{n,m,x}} \mathbf{Q}, M_x)$. For the B -part, the action of \mathcal{A}_m on $\mathbf{R}f_*M_B$ is given by the monodromy action.*

4.4. Comparison to de Rham cohomologies and chain complexes.

4.4.1. *Comparison to de Rham complexes.* We show that the B -part is equal to Gauss-Manin connection with the coefficient in M_{dR} . If $m = n - 1$, then using the commutation relation, $\mathcal{A}_{n,dR}$ can be written as the formal power series ring.

$$\mathcal{A}_{n,dR} = \mathcal{A}_{n-1,dR} \langle \langle t_{n,1}, \dots, t_{n,n-2} \rangle \rangle$$

as a vector space. The multiplication rule is given by the commutation relation. Let M_{dR} be a continuous $\mathcal{A}_{n,dR}$ -module. Then the action of t_{ij} gives a nilpotent endomorphism E_{ij} on M_{dR} .

Proposition 4.6. *As a vector space $\mathbf{R}f_*M_{dR}$ is quasi-isomorphic to*

$$\mathbf{R}f'_*M_{dR} : M_{dR} \xrightarrow{\nabla} M_{dR} \otimes \Omega_{n/m}^1 \xrightarrow{\nabla} M_{dR} \otimes \Omega_{n/m}^2 \rightarrow \dots$$

$$\mathbf{R}f''_*M_{dR} : M_{dR} \xrightarrow{\nabla} M_{dR} \otimes \Omega_{\mathcal{M}_{0,n}/\mathcal{M}_{0,m}}^1 \xrightarrow{\nabla} M_{dR} \otimes \Omega_{\mathcal{M}_{0,n}/\mathcal{M}_{0,m}}^2 \rightarrow \dots$$

Here $\Omega_{n/m}^\bullet$ is a subcomplex of the relative de Rham complex $\Omega_{\mathcal{M}_{0,n}/\mathcal{M}_{0,m}}^\bullet$ generated by $\frac{d(x_i - x_j)}{x_i - x_j}$. As a consequence,

- (1) if M is finite dimensional, then $\mathbf{R}^i f_*M_{dR}$ is also finite dimensional, and
- (2) $\mathbf{R}^i f_*M_{dR} = 0$ if $i > n - m$.

Proof. Since the action of $\langle E_{ij} \rangle$ are nilpotent, we can show that $\mathbf{R}f'_*M_{dR}$ and $\mathbf{R}f''_*M_{dR}$ are quasi-isomorphic by the induction of the length of nilpotent filtrations. \square

To give an explicit quasi-isomorphism, it is convenient to introduce the bar complex. Let $\overline{B_n}$ be the reduced bar complex of logarithmic bar complex Ω_n^\bullet . Then the topological dual of \mathcal{A}_n is isomorphic $B_n = H^0(\overline{B_n}) \subset \overline{B_n}$ and $H^i(B_n) = 0$ for $i \neq 0$. Then B_n becomes a Hopf algebra and the \mathcal{A}_n action on M_{dR} yields a right B_n -comodule structure on M_{dR} . By the definition (4.1), $\mathbf{R}f_*M_{dR}$ is equal to

$$0 \xrightarrow{d_0} M_{dR} \otimes B_m \xrightarrow{d_1} M_{dR} \otimes B_n \otimes B_m \rightarrow M_{dR} \otimes B_n \otimes B_m \otimes B_m \rightarrow \dots$$

For example d_0, d_1 is given by the formula

$$\begin{aligned} d_0(a \otimes m) &= \Delta_m(a) \otimes m - a \otimes \Delta_M(m) \\ d_1(a \otimes b \otimes m) &= \Delta_m(a) \otimes b \otimes m - a \otimes \Delta(b) \otimes m + a \otimes b \otimes \Delta_M(m) \end{aligned}$$

Here $\Delta_m : B_m \rightarrow B_m \otimes B_n$, $\Delta : B_n \rightarrow B_n \otimes B_n$ and $\Delta_M : M_{dR} \rightarrow B_n \otimes M_{dR}$ are the coproducts.

Proposition 4.7. (1) Let $\psi^k : M_{dR} \otimes B_n^{\otimes k} \otimes B_m \rightarrow M_{dR} \otimes \Omega_{n/m}$ be a map defined by

$$m \otimes a_1 \otimes \cdots \otimes a_k \otimes b \mapsto m \otimes \pi(a_1) \cdots \pi_k(a_1)\epsilon(b),$$

where $\epsilon : B_m \rightarrow \mathbf{Q}$ is the augmentation. Then $\sum_k \psi^k$ is a homomorphism of complex and quasi-isomorphism.

(2) The action of $\mathcal{A}_{m,dR}$ on $\mathbf{R}^i f_* M_{dR}$ is equal to Gauss-Manin connection.

4.4.2. *Comparison to chain complex.* Let M be an $\mathcal{A}_n^\Phi[[\alpha_i]]$ module and $M^* = \text{Hom}_{\mathbf{Q}[[\alpha_i]]}(M, \mathbf{Q}[[\alpha_i]]^\Phi)$. Then M_B and M_B^* define local systems on \mathcal{M}_n . The homology and the cohomology in the coefficient in M_B^* and M_B is denoted by $H_i^B(\mathcal{M}_n, M_B^*)$ and $H_B^i(\mathcal{M}_n, M_B)$, respectively. We have the natural pairing

$$H_i^B(\mathcal{M}_n, M_B^*) \otimes H_B^i(\mathcal{M}_n, M_B) \rightarrow \mathbf{C}[[\alpha_i]]$$

and via this map we have the following evaluation homomorphism

$$ev : H_i^B(\mathcal{M}_n, M_B^*) \rightarrow H_B^i(\mathcal{M}_n, M_B)^*.$$

4.5. **Φ -integral.** We have an isomorphism

$$H^i(\mathcal{M}_n, M)_{dR} \simeq H_{dR}^i(\mathcal{M}_n, M_{dR})$$

and

$$H^i(\mathcal{M}_n, M)_B \simeq H_B^i(\mathcal{M}_n, M_B).$$

The homology $H_i^B(\mathcal{M}_n, M^*)$ is identified with the homology group of chains complex with the coefficient in M^* . An element σ of the chain complex is a linear combination of $[\gamma, f]$ where γ is an i -chain in \mathcal{M}_n and f is a section of M^* on γ .

Definition 4.8 (Φ -integral, twisted chain). (1) Let $\sigma = [\gamma, f] \in H_i^B(\mathcal{M}_n, M_B^*)$ and $\omega \in H_{dR}^i(\mathcal{M}_n, M_{dR})$. We define a Φ -integral by

$$\int_\gamma^\Phi f \omega = ev(\sigma)(c_H^{-1}(\omega)) \in \mathbf{C}[[\alpha_i]]$$

Φ -integral defines a pairing

$$H_i^B(\mathcal{M}_n, M_B^*) \otimes H_{dR}^i(\mathcal{M}_n, M_{dR}) \rightarrow \mathbf{C}[[\alpha_i]]$$

(2) Let φ_i ($i = 1, \dots, l$) be admissible functions on \mathcal{M}_n , D a domain defined by $0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-3} \leq 1$ for some distinguished coordinates x_1, \dots, x_{n-3} . Assume that the values of φ_i are positive and real on D . The twisted chain on D with the product of positive real branches of $\varphi_i^{\alpha_i}$ is denoted by $\prod_{i=1}^l \varphi_D^{\alpha_i}$.

4.6. Regularization of cycles and specialization of exponents. M be an \mathcal{A}_4^Φ -module and M^* its dual. Then

$$\begin{aligned} Mx^\alpha(1-x)^\beta &= M \otimes x^\alpha(1-x)^\beta \mathbf{Q}((\beta))[[\alpha]] \\ M^*x^{-\alpha}(1-x)^{-\beta} &= M \otimes x^{-\alpha}(1-x)^{-\beta} \mathbf{Q}((\beta))[[\alpha]] \end{aligned}$$

becomes an $\mathcal{A}_4^\Phi((\beta))[[\alpha]]$ -module. Let $v \in M_{B,\overline{01}}^*$ be an element invariant under the action of local monodromy at $\overline{01}$ and $w \in M_{dR}$. By the standard argument for the regularization of topological cycles around 0 and 1, we have the following proposition.

Proposition 4.9. *The extension $\tilde{v}(\alpha)$ to $[0, 1]$ of the element $v\alpha x^\alpha(1-x)^\beta$ defines a homology class in $H_1(\mathcal{M}_4, M^*x^\alpha(1-x)^\beta)$.*

Proof. The action of the local monodromy ρ_0 at 0 on $vx^\alpha(1-x)^\beta$ is given by

$$vx^\alpha(1-x)^\beta \rightarrow \mathbf{e}(\alpha)vx^\alpha(1-x)^\beta.$$

Therefore

$$\alpha vx^\alpha(1-x)^\beta = \frac{\alpha}{\mathbf{e}(\alpha)-1}(\rho_0 - 1)vx^\alpha(1-x)^\beta = \frac{\alpha}{\mathbf{e}(\alpha)-1}\partial(vx^\alpha(1-x)^\beta|_{c_0})$$

where c_0 is the small circle around 0. The action of $\rho_1 - 1$ is invertible where ρ_1 is the local monodromy. Therefore $v\alpha x^\alpha(1-x)^\beta$ defines a homology class in $H_1(\mathcal{M}_4, M^*x^\alpha(1-x)^\beta)$. \square

Definition 4.10. *The extension of $\alpha vx^\alpha(1-x)^\beta$ is called the regularized cycle.*

As a consequence of the above proposition, we have a pairing

$$F(\alpha) = (\tilde{v}(\alpha), \omega \frac{dx}{x}) \in \mathbf{C}((\beta))[[\alpha]].$$

The specialization $\tilde{v}(0)$ of $\tilde{v}(\alpha)$ defines an element in $H_1(\mathcal{M}_4, M^*(1-x)^\beta)$. The specialization $F(0) \in \mathbf{Q}((\beta, \gamma))$ is equal to the pairing $(\tilde{v}(0), \omega \frac{dx}{x})$ between $H_1(\mathcal{M}_4, M^*(1-x)^\beta)$ and $H_{dR}^1(\mathcal{M}_4, M(1-x)^{-\beta})$. Since the monodromy action on $\tilde{v}(0)$ around 0 is trivial, we have the following proposition.

Proposition 4.11. *$\tilde{v}(0)$ is contained in the image of the map*

$$H_1(\Delta_0^*, M^*(1-x)^\beta) \rightarrow H_1(\mathcal{M}_4, M^*(1-x)^\beta),$$

where Δ_0^* is the small disc around 0. As a consequence, we have

$$(\tilde{v}(0), \omega \frac{dx}{x}) = (v, \omega)$$

Proof. $\tilde{v}(\alpha)$ is sum of $\alpha vx^\alpha(1-x)^\beta$ and $\frac{\alpha}{\mathbf{e}(\alpha)-1}vx^\alpha(1-x)^\beta|_{c_0}$. By taking the limit for $\alpha \rightarrow 0$, it tends to $vx^\alpha(1-x)^\beta|_{c_0}$. \square

Remark 4.12. *In classical case, the function $\alpha x^{\alpha-1}$ on $[0, 1]$ tends to the delta function supported at 0 when α tends to zero. The above proposition is reinterpretation of this fact using regularization of topological cycles.*

Definition 4.13. (1) For a section v of M on $[0, 1]$, the pairing of $\omega \in H_{dR}^1(\mathcal{M}_4, Mx^{-\alpha}(1-x)^{-\beta})$ and the regularized cycle of $vx^\alpha(1-x)^\beta$ in $H_1^B(\mathcal{M}_4, M^*x^\alpha(1-x)^\beta)$ is denoted by

$$\int_{[0,1]}^\Phi vx^\alpha(1-x)^\beta \omega$$

(2) For a section v of M on $[0, 1]$, the relative cycle modulo $\{\overline{01} \cup \overline{10}\}$ defined by v is denoted by $v_{(0,1)}$. For an element

$$\omega \in H_{dR}^1(\mathcal{M}_4, Mx^{-\alpha}(1-x)^{-\beta} (\text{mod } \overline{01} \cup \overline{10}))$$

the paring of $v_{(0,1)}$ and ω is denoted by $\int_{(0,1)}^\Phi v\omega$.

The following proposition is direct from the definition.

Proposition 4.14. Let u_0, u_1 be elements in M_{dR} . Then

$$\begin{aligned} & \int_{[0,1]} vx^\alpha(1-x)^\beta (u_0 \frac{dx}{x} + u_1 \frac{dx}{x-1}) - \int_{(0,1)} vx^\alpha(1-x)^\beta (u_0 \frac{dx}{x} + u_1 \frac{dx}{x-1}) \\ &= ((R_0 + \alpha I_M)^{-1} v(\overline{01}), u_0) + ((R_1 + \beta I_M)^{-1} v(\overline{10}), u_1) \end{aligned}$$

5. Φ -BETA MODULE AND Φ -HYPERGOEMTRIC MODULES

In this section, we define a \mathcal{A}_4^Φ -module associated to beta function and hypergeometric functions. We fix an associator Φ throughout this section.

5.1. Beta module and 1-cocycle relation.

Definition 5.1. (1) We set $\mathcal{F}(\chi) = x^\alpha(1-x)^\beta \mathbf{Q}[[\alpha, \beta]]$. We define the pre-beta module $\mathbf{B}_\Phi^*(\alpha, \beta)$ by

$$\mathbf{B}_\Phi^*(\alpha, \beta) = H_\Phi^1(\mathcal{M}_{0,4}, \mathcal{F}(\chi)).$$

It is a $\mathbf{Q}[[\alpha, \beta]]^\Phi$ -module in \mathcal{C} .

(2) The sub modules

$$\mathbf{B}_{\Phi,B}(\alpha, \beta) = 2\pi i \alpha \cdot \mathbf{B}_{\Phi,B}^*(\alpha, \beta) \cap 2\pi i \beta \cdot \mathbf{B}_{\Phi,B}^*(\alpha, \beta) \subset \mathbf{B}_{\Phi,B}^*(\alpha, \beta)$$

$$\mathbf{B}_{\Phi,dR}(\alpha, \beta) = \alpha \cdot \mathbf{B}_{\Phi,dR}^*(\alpha, \beta) \cap \beta \cdot \mathbf{B}_{\Phi,dR}^*(\alpha, \beta) \subset \mathbf{B}_{\Phi,dR}^*(\alpha, \beta)$$

defines a sub object in $\mathbf{B}_\Phi^*(\alpha, \beta)$, which is called the Beta module.

$\mathbf{B}_\Phi(\alpha, \beta)_B$ and (resp. $\mathbf{B}_\Phi(\alpha, \beta)_{dR}$) is a free $\mathbf{Q}[[\alpha, \beta]]_B$ (resp. $\mathbf{Q}[[\alpha, \beta]]_{dR}$) modules of rank one generated by φ characterized by $\varphi(x^\alpha(1-x)_{[0,1]}^\beta) = 1$ (resp. $\alpha \frac{dx}{x}$). We define modified Φ -beta function $B'_\Phi(\alpha, \beta)$ by

$$c_4(\varphi) B'_\Phi(\alpha, \beta) = \alpha \frac{dx}{x}$$

In other words,

$$B'_\Phi(\alpha, \beta) = \int_{[0,1]}^\Phi x^\alpha(1-x)^\beta \alpha \frac{dx}{x}$$

Since

$$x^\alpha(1-x)^\beta\left(\alpha\frac{dx}{x} + \beta\frac{dx}{1-x}\right) = 0$$

in $H^1(\mathcal{M}_4, \mathcal{F}_{\overline{01}*}(\chi))$, we have $B_\Phi(\alpha, \beta) = B_\Phi(\beta, \alpha)$.

Theorem 5.2. *We have the following one cocycle relation for modified Φ -beta functions.*

$$B'_\Phi(\alpha, \gamma + \beta)B'_\Phi(\gamma, \beta) = B'_\Phi(\alpha, \gamma)B'_\Phi(\alpha + \gamma, \beta)$$

As a consequence,

$$B'_\Phi(\alpha, \beta) = \exp\left(\sum_{i \geq 2} a_n(\alpha^n + \beta^n + (-\alpha - \beta)^n)\right)$$

for some $a_n = a_{n,\Phi} \in \mathbf{C}$.

Before proving the theorem, we define the \mathcal{A}_5^Φ module \mathcal{F} . We define $\mathcal{F} = (1-x)^\alpha y^\beta (x-y)^\gamma \mathbf{Q}[[\alpha, \beta, \gamma]]$ by taking a coordinate $p = (x, y, 0, 1, \infty)$ of \mathcal{M}_5 . We set $D = \{0 \leq y \leq x \leq 1, x, y \in \mathbf{R}\}$. We consider another coordinates $p = (1, \eta, 0, \xi, \infty)$ of \mathcal{M}_4 . Then we have relations $\eta = \frac{y}{x}, \xi = \frac{1}{x}$. We define homomorphisms $f_1, f_2 : \mathcal{M}_4$ by $f_1(p) = x, f_2(p) = \eta$ and (f_1, f_2) by the composite

$$\mathcal{M}_5 \xrightarrow{\Delta} \mathcal{M}_5 \times \mathcal{M}_5 \xrightarrow{f_1 \times f_2} \mathcal{M}_4 \times \mathcal{M}_4.$$

Thus we get a homomorphism of algebroid

$$(f_1, f_2)_* : \mathcal{A}_5^\Phi \rightarrow \mathcal{A}_4^\Phi \otimes \mathcal{A}_4^\Phi$$

and its abelianization $\mathcal{A}_5^{\Phi,ab} \rightarrow \mathcal{A}_4^{\Phi,ab} \otimes \mathcal{A}_4^{\Phi,ab}$. Since

$$(1-x)^\alpha y^\beta (x-y)^\gamma = (1-x)^\alpha x^{\beta+\gamma} \eta^\beta (1-\eta)^\gamma$$

we have

$$\begin{aligned} \mathcal{F} &= (f_1, f_2)^*((1-x)^\alpha x^{\beta+\gamma} \eta^\beta (1-\eta)^\gamma \mathbf{Q}[[\alpha, \beta, \gamma]]) \\ &= (f_1, f_2)^*((1-x)^\alpha x^{\beta+\gamma} \mathbf{Q}[[\alpha, \beta + \gamma]] \\ &\quad \otimes_{\mathbf{Q}[[\beta+\gamma]]} \eta^\beta (1-\eta)^\gamma \mathbf{Q}[[\beta, \gamma]]). \end{aligned}$$

Therefore we have a homomorphism

$$(f_1, f_2)^* : \mathbf{B}_\Phi(\alpha, \beta + \gamma) \otimes_{\mathbf{Q}[[\beta+\gamma]]} \mathbf{B}_\Phi(\beta, \gamma) \rightarrow H_\Phi^2(\mathcal{M}_5, \mathcal{F}).$$

Lemma 5.3. (1)

$$\begin{aligned} &(f_1, f_2)^*\left((1-x)^\alpha x^{\beta+\gamma} \eta^\beta (1-\eta)^\gamma \frac{(\beta+\gamma)dx}{x} \wedge \frac{\beta d\eta}{\eta}\right) \\ &= (1-x)^\alpha y^\beta (x-y)^\gamma \alpha \beta \frac{dx}{x-1} \wedge \frac{dy}{y} \end{aligned}$$

$$(2) \quad (f_1, f_2)_*(D) = [0, 1] \times [0, 1] \text{ in } H_{dR}^2(\mathcal{M}_5, \mathcal{F}_{dR}).$$

Proof. Since the element

$$(1-x)^\alpha y^\beta (x-y)^\gamma \left(\alpha \frac{dx}{x-1} + \beta \frac{dy}{y} + \gamma \frac{d(x-y)}{x-y}\right)$$

is exact, we have the equality in $H_{dR}^2(\mathcal{M}_5, \mathcal{F}_{dR})$. \square

Proof of Theorem 5.2. By Lemma 5.3, we have

$$\begin{aligned} & \int_D^\Phi (1-x)^\alpha y^\beta (x-y)^\gamma \alpha \beta \frac{dx}{x-1} \wedge \frac{dy}{y} \\ &= \int_{[0,1]}^\Phi (1-x)^\alpha x^{\beta+\gamma} \frac{(\beta+\gamma)dx}{x} \int_{[0,1]}^\Phi \eta^\beta (1-\eta)^\gamma \frac{\beta d\eta}{\eta} \\ &= B'_\Phi(\alpha, \gamma + \beta) \cdot B'_\Phi(\gamma, \beta) \end{aligned}$$

Since the first integral is symmetric on α and β , we have

$$B'_\Phi(\alpha, \gamma + \beta) \cdot B'_\Phi(\gamma, \beta) = B'_\Phi(\beta, \gamma + \alpha) \cdot B'_\Phi(\gamma, \alpha).$$

□

Definition 5.4. (1) We define the Beta function $B(\alpha, \beta)$ by

$$B_\Phi(\alpha, \beta) = B'_\Phi(\alpha, \beta) \frac{\alpha + \beta}{\alpha \beta}$$

(2) We define $\Gamma_\Phi \in \mathbf{C}((x))$ by

$$\Gamma_\Phi(x) = \frac{1}{x} \exp\left(\sum_{i=2}^{\infty} a_n x^n\right).$$

Here $a_n \in \mathbf{C}$ is defined in Theorem 5.2.

(3) The product $\Gamma_\Phi(a_1) \cdots \Gamma_\Phi(a_n)$ is denoted by $\Gamma_\Phi(a_1, \dots, a_n)$.

By the definition of Φ -Gamma function and Proposition 5.2, we have

$$B_\Phi(\alpha, \beta) = \frac{\Gamma_\Phi(\alpha)\Gamma_\Phi(\beta)}{\Gamma_\Phi(\alpha + \beta)}$$

5.2. Definition of Φ -hypergeometric modules. In this section, we define a framed \mathcal{A}_4^Φ -modules at $\overline{01}$ associated to generalized hypergeometric functions.

Lemma 5.5. Let x_1, \dots, x_k be the distinguished coordinates of \mathcal{M}_{k+3} . We set $\xi_1 = x_1, \xi_2 = \frac{x_2}{x_1}, \dots, \xi_i = \frac{x_k}{x_{k-1}}$. Then

- (1) ξ_1, \dots, ξ_k and $1 - \xi_1, \dots, 1 - \xi_k$ and $1 - \xi_1 \xi_2 \cdots \xi_k$ are admissible functions of \mathcal{M}_{k+3} .
- (2) $\frac{d\xi_1}{\xi_1}, \dots, \frac{d\xi_k}{\xi_k}$ are admissible one forms.

We consider $\mathcal{M}_{0,5}$ and $\mathcal{M}_{0,6}$ with the distinguished coordinates (x_1, t) and (x_1, x_2, t) . We define π_5, π_6 by

$$\begin{aligned} \pi_5 : \mathcal{M}_{0,5} &\rightarrow \mathcal{M}_{0,4} : (x_1, t) \mapsto t \\ \pi_6 : \mathcal{M}_{0,6} &\rightarrow \mathcal{M}_{0,4} : (x_1, x_2, t) \mapsto t. \end{aligned}$$

We define cell's D_5 and D_6 in $\mathcal{M}_{0,5}(\mathbf{R})_{\overline{01}}$ and $\mathcal{M}_{0,6}(\mathbf{R})_{\overline{01}}$ by

$$\begin{aligned} D_5 &= \{\xi_1 \in [0, 1], t = \overline{01}\}, \\ D_6 &= \{\xi_1, \xi_2 \in [0, 1] \leq 1, t = \overline{01}\}, \end{aligned}$$

We define \mathcal{A}_5^Φ and \mathcal{A}_6^Φ modules $\mathcal{F}(a_1, a_2; b_1)$ and $\mathcal{F}(a_1, a_2, a_3; b_1, b_2)$ by

$$\xi_1^{a_1} (1 - \xi_1)^{b_1 - a_1} (1 - t\xi_1)^{-a_2} \mathbf{Q}[[a_1, a_2, b_1]].$$

and

$$\xi_1^{a_1} \xi_2^{a_2} (1 - \xi_1)^{b_1 - a_1} (1 - \xi_2)^{b_2 - a_2} (1 - t\xi_1 \xi_2)^{-a_3} \mathbf{Q}[[a_1, a_2, a_3, b_1, b_2]].$$

Definition 5.6. We use the notation of cycles $\gamma_1, \gamma_2, \gamma_1^\#, \gamma_2^\#$ defined in (2.7).

(1) We define hypergeometric modules $HM(a_1, a_2, b_1)$ and $HM(a_1, a_2, a_3, b_1, b_2)$ on \mathcal{A}_4^Φ by

$$HM(a_1, a_2, b_1) = \mathbf{R}^1 \pi_{5*} \mathcal{F}(a_1, a_2, b_1) \otimes \mathbf{B}_\Phi(a_1, b_1 - a_1)^{-1}$$

$$HM(a_1, a_2, a_3, b_1, b_2) = \mathbf{R}^2 \pi_{6*} \mathcal{F}(a_1, a_2, a_3, b_1, b_2) \\ \otimes \mathbf{B}_\Phi(a_1, b_1 - a_1)^{-1} \otimes \mathbf{B}_\Phi(a_2, b_2 - a_2)^{-1}$$

(2) We define hypergeometric function

$$F_\Phi : \mathcal{A}_{4, B, \overline{01}*} \rightarrow \mathbf{C}[[a_1, a_2, b_1]]$$

by

$$F_\Phi(a_1, a_2, b_1 + 1, \gamma)$$

$$= B_\Phi(a_1, b_1 - a_1 + 1)^{-1} \int_{\gamma(\gamma_1)}^\Phi \xi_1^{a_1} (1 - \xi_1)^{b_1 - a_1} (1 - t\xi_1)^{-a_2} \frac{d\xi_1}{\xi_1}$$

for $\gamma \in \mathcal{A}_{4, \overline{01}*}$.

Proposition 5.7.

$$(1 \quad 0) \varphi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B_\Phi(a, c - a + 1)^{-1} \int_{\varphi(D_5)}^\Phi \xi_1^a (1 - \xi_1)^{c-a} (1 - t\xi_1)^{-b} \frac{d\xi_1}{\xi_1}$$

$$(0 \quad 1) \varphi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B_\Phi(a, c - a + 1)^{-1} \\ \frac{b}{a} \int_{\varphi(D_5)}^\Phi t\xi_1^{a+1} (1 - \xi_1)^{c-a} (1 - t\xi_1)^{-b-1} \frac{d\xi_1}{\xi_1}$$

Proof. The regularized cycle D_5 defines an element in $HM_{B, \overline{01}}^*$. By choosing the base (2.8), we have

$$\int_{[0,1] \times \overline{01}}^\Phi \omega_2 = 0,$$

$$\int_{[0,1] \times \overline{01}}^\Phi \omega_1 = \int_{[0,1]}^\Phi \xi_1^a (1 - \xi_1)^{c-a} (1 - t\xi_1)^{-b} \frac{d\xi_1}{\xi_1} \Big|_{t=\overline{01}} = B_\Phi(a, c - a + 1).$$

and as a consequence, we have

$$c_{HM^*}(\gamma_1(\overline{01})) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} B_\Phi(a, c - a + 1).$$

□

Proposition 5.8. *Let $\gamma_1^\#, \gamma_2^\#$ be the cycles defined in (2.7). Then*

$$\begin{aligned} c_{HM}(\gamma_1^\#(\overline{10})) &= \left(\frac{1}{\frac{-b}{a+b-c}} \right) B_\Phi(a, b-c), \\ c_{HM}(\gamma_2^\#(\overline{10})) &= \left(\frac{0}{\frac{a+b-c-1}{a}} \right) B_\Phi(c+1-a, 1-b). \end{aligned}$$

Proposition 5.9. *Let Φ be the associator and Φ_0, Φ_1 be elements in $\mathbf{C}[[e_0, e_1]]$ defined by*

$$\Phi(e_0, e_1) = 1 + \Phi_0(e_0, e_1)e_0 + \Phi_1(e_0, e_1)e_1$$

Then

$$B'_\Phi(-a, -b) = \Phi_1^{ab}(a, b)b + 1$$

where $\Phi_0^{ab}(a, b)$ is the image under the abelianization map $\mathbf{C}\langle\langle e_0, e_1 \rangle\rangle \rightarrow \mathbf{C}[[\alpha, \beta]]$. As a consequence, we have $a_n = \zeta^\Phi(n)$.

Proof. By Proposition 5.7, Proposition 5.8, and relation (2.7), we have

$$\begin{aligned} &(0 \quad 1) \Phi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= B_\Phi(a, c-a+1)^{-1} \frac{b}{a} \int_{\gamma_1}^{\Phi} t\xi_1^{a+1} (1-\xi_1)^{c-a} (1-t\xi_1)^{-b-1} \frac{d\xi_1}{\xi_1} \Big|_{t=\overline{01}} \\ &= B_\Phi(a, c-a+1)^{-1} \left(\frac{\mathbf{s}(b-c)}{\mathbf{s}(a+b-c)} \frac{-b}{a+b-c} B_\Phi(a, b-c) \right. \\ &\quad \left. + \frac{\mathbf{s}(b)}{\mathbf{s}(c-a-b)} \frac{a+b-c-1}{a} B_\Phi(c+1-a, 1-b) \right) \end{aligned}$$

We take a limit for $a \rightarrow 0$ and apply Proposition 3.17. Then we have

$$\begin{aligned} (-b)\Phi_1^{ab}(-c, c-b) &= \lim_{a \rightarrow 0} (1 \quad 0) \Phi(P_0, P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{b}{b-c} \left(-1 + \frac{\Gamma_\Phi(b-c+1, c+1)}{\Gamma_\Phi(b+1)} \right), \end{aligned}$$

and

$$\Phi^{ab}(-c, c-b)(c-b) + 1 = \frac{\Gamma_\Phi(b-c+1, c+1)}{\Gamma_\Phi(b+1)}.$$

□

5.3. Junction.

5.3.1. Definition of Junctions. Let M, N be \mathcal{A}_4 modules. Let $pr_2 : \mathcal{M}_4 \times \mathcal{M}_4 \rightarrow \mathcal{M}_4 : (x, y) \mapsto y$ be the second projection, $\Delta : \mathcal{M}_4 \rightarrow \mathcal{M}_4 \times \mathcal{M}_4$ be the diagonal map, $i : \mathcal{M}_4 \rightarrow \mathcal{M}_4 \times \mathcal{M}_4$ be an infinitesimal inclusion defined by $x \mapsto (x, \overline{01})$. We consider the map

$$\begin{aligned} \alpha : pr_2^* M \otimes pr_1^*(M^* \otimes N) &\rightarrow i^*(pr_2^* M \otimes pr_1^*(M^* \otimes N)) \simeq M \otimes (M^* \otimes N)_{\overline{01}} \\ \beta : pr_2^* M \otimes pr_1^*(M^* \otimes N) &\rightarrow \Delta^*(pr_2^* M \otimes pr_1^*(M^* \otimes N)) \simeq M \otimes M^* \otimes N \xrightarrow{ev} N \end{aligned}$$

Here $ev : M \otimes M^* \otimes N \rightarrow N$ is given by the evaluation map. Then we have a complex $\mathbf{E}(M, N)$:

$$\mathbf{E}(M, N) : pr_2^*M \otimes pr_1^*(M^* \otimes N) \xrightarrow{\alpha \oplus \beta} i_*(M \otimes (M^* \otimes N)_{\overline{01}}) \oplus \Delta_*N$$

We define the junction $E(M, N) = \mathbf{R}^1pr_{2*}\mathbf{E}(M, N)$

5.3.2. Gauss-Manin connection for a junction. We consider the Gauss-Manin connection on $E(M, N)_{dR}$. Let

$$\begin{aligned} P &= P_0 \frac{dx}{x} + P_1 \frac{dx}{x-1} : M_{dR} \rightarrow M_{dR} \otimes \left\langle \frac{dx}{x}, \frac{dx}{x-1} \right\rangle \\ Q &= Q_0 \frac{dx}{x} + Q_1 \frac{dx}{x-1} : N_{dR} \rightarrow N_{dR} \otimes \left\langle \frac{dx}{x}, \frac{dx}{x-1} \right\rangle \end{aligned}$$

be the associated connection of M and N . By choosing basis $\{\omega_i\}$ and $\{\eta_j\}$ of M_{dR} and N_{dR} , the map P_0, \dots, Q_1 can be expressed as matrices by the rule:

$$(5.1) \quad \nabla \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = P \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

The module $\mathbf{E}(M, N)_{dR}$ is quasi-isomorphic to the associate simple complex of the following complex \mathbf{E}_{dR}

$$\begin{array}{ccc} M_{dR} \otimes M_{dR}^* \otimes N_{dR} & \xrightarrow{-1 \otimes {}^tP \otimes 1 + 1 \otimes 1 \otimes Q} & M_{dR} \otimes M_{dR}^* \otimes N_{dR} \otimes \left\langle \frac{dx}{x}, \frac{dx}{x-1} \right\rangle \\ \downarrow id \oplus ev & & \\ (M_{dR} \otimes M_{dR}^* \otimes N_{dR}) & & \\ \oplus N_{dR} & & \end{array}$$

Therefore

$$(5.2) \quad \begin{aligned} H^1(\mathbf{E}_{dR}) \\ \simeq M_{dR} \otimes M_{dR}^* \otimes N_{dR} \frac{dx}{x} \oplus M_{dR} \otimes M_{dR}^* \otimes N_{dR} \frac{dx}{x-1} \oplus N_{dR} \end{aligned}$$

Under this isomorphism, the Gauss-Manin connection

$$\nabla : H^1(\mathbf{E}_{dR}) \rightarrow H^1(\mathbf{E}_{dR}) \otimes \left\langle \frac{dy}{y}, \frac{dy}{y-1} \right\rangle$$

is given by

$$(5.3) \quad \begin{aligned} &\nabla(u_0 \frac{dx}{x} + u_1 \frac{dx}{x-1} + v) \\ &= ((P \otimes 1 \otimes 1)(u_0) \frac{dx}{x} + (P \otimes 1 \otimes 1)(u_1) \frac{dx}{x-1} + ev(u_0) \frac{dy}{y} + ev(u_1) \frac{dy}{y-1} + Q(v)) \\ &= \left((P_0 \otimes 1 \otimes 1)(u_0) \frac{dx}{x} + (P_0 \otimes 1 \otimes 1)(u_1) \frac{dx}{x-1} + ev(u_0) + Q_0(v) \right) \frac{dy}{y} \\ &\quad + \left((P_1 \otimes 1 \otimes 1)(u_0) \frac{dx}{x} + (P_1 \otimes 1 \otimes 1)(u_1) \frac{dx}{x-1} + ev(u_1) + Q_0(v) \right) \frac{dy}{y-1} \end{aligned}$$

for $u_0, u_1 \in M_{dR} \otimes M_{dR}^* \otimes N_{dR}$ and $v \in N_{dR}$.

Bases of $M_{dR} \otimes M_{dR}^* \otimes N_{dR} \frac{dx}{x}$, $M_{dR} \otimes M_{dR}^* \otimes N_{dR} \frac{dx}{x-1}$, and N_{dR} form a basis of $E(M, N)_{dR}$. Using this basis, the connection ∇ on $E(M, N)_{dR}$ can be expressed as $R_0 \frac{dy}{y} + R_1 \frac{dy}{y-1}$ via the rule (5.1), where

$$R_0 = \begin{pmatrix} P_0 \otimes 1 \otimes 1 & 0 & Ev \\ 0 & P_0 \otimes 1 \otimes 1 & 0 \\ 0 & 0 & Q_0 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} P_1 \otimes 1 \otimes 1 & 0 & 0 \\ 0 & P_1 \otimes 1 \otimes 1 & Ev \\ 0 & 0 & Q_1 \end{pmatrix}.$$

by the formula (5.3).

5.3.3. Horizontal section of the dual. The action of $\varphi(e_0, e_1) \in \mathcal{A}_{4,dR} = \mathbf{C}\langle\langle e_0, e_1 \rangle\rangle$ on $H^1(E_{dR})^*$ is given by the left multiplication of the matrix $\varphi(R_0, R_1)$. For $I = (i_1, \dots, i_n) \in \{0, 1\}^n$, we set

$$P_I = P_{i_1} \cdots P_{i_n}, \quad R_I = R_{i_1} \cdots R_{i_n}, \text{etc}$$

By (5.1), we have the following proposition.

Proposition 5.10. *We have*

$$R_I(v^*) = \sum_{I=J_1 0 J_2} ((P_{J_1} \otimes 1 \otimes 1)Ev(Q_{J_2}(v^*)) \frac{dx^*}{x} +$$

$$+ \sum_{I=J_1 1 J_2} ((P_{J_1} \otimes 1 \otimes 1)Ev(Q_{J_2}(v^*)) \frac{dx^*}{x-1} + Q_I(v^*)$$

for $u_0^*, u_1^* \in M_{dR}^* \otimes M_{dR} \otimes N_{dR}^*$ and $v^* \in N_{dR}^*$.

As a consequence, for φ given in (2.11), we have

$$\varphi(R_0, R_1)(v^*) = \sum_{J_1, J_2} c_{J_1 0 J_2} ((P_{J_1} \otimes 1 \otimes 1)Ev(Q_{J_2}(v^*)) \frac{dx^*}{x} +$$

$$+ \sum_{J_1, J_2} c_{J_1 1 J_2} ((P_{J_1} \otimes 1 \otimes 1)Ev(Q_{J_2}(v^*)) \frac{dx^*}{x-1} + \varphi(Q_0, Q_1)(v^*)$$

5.3.4. Betti part of the dual. Using the chain complex the dual local sysstem of the Betti-part $(E(M, N)_B)^*$ of the junction $E(M, N)$ is naturally isomorphic to the cohomology of the associate simple complex of the following chain complex.

$$C_\bullet(\mathcal{M}_4/\mathcal{M}_4, M_y^* \otimes M_{\overline{01}} \otimes N_{\overline{01}}^*) \xrightarrow{i_* \oplus ev_*} C_\bullet(\mathcal{M}_5/\mathcal{M}_4, M_y^* \otimes M_x \otimes N_x^*)$$

$$\oplus C_\bullet(\mathcal{M}_4/\mathcal{M}_4, N_x^*)$$

Definition 5.11. Let $y \in [0, 1], \tau \in N_y^*$ and $\Delta \in M_y^* \otimes M_y$ be the element corresponding to the identity element. The local section of $pr_1^* M^* \otimes pr_2^*(M \otimes$

$N)$ on $\{(x, y) \mid 0 \leq x \leq y\}$ whose fiber at (y, y) is equal to $\Delta \otimes \tau \in M_y^* \otimes M_y \otimes N_y^*$ to is denoted by $\delta(\tau)$. The element

$$\begin{aligned} & \delta(\tau)_{\{x|0 \leq x \leq y\} \times \{y\}} + \tau_{(y,y)} - \delta(\tau)_{(\overline{01},y)} \\ & \in C_1(pr_2^{-1}(y), M_y^* \otimes M \otimes N^*) \oplus C_0(y \times y, N_y^*) \oplus C_0(\overline{01} \times y, M_y^* \otimes M_{\overline{01}} \otimes N_{\overline{01}}^*) \end{aligned}$$

is closed and defines an element $J(y, \tau)$ of $E(M, N)_{B,y}^*$, which is called the junction cycle for τ .

If $\tau_{\overline{01}}$ and $\tau_{\overline{10}}$ are fibers of a local section τ of N^* on $[0, 1]$, then we have $[0, 1]J(\overline{01}, \tau_{\overline{01}}) = J(\overline{10}, \tau_{\overline{10}})$ using the action of $\mathcal{A}_{4,B}$. In this situation, $J(y, \tau_y)$ is denoted by $J(y, \tau)$.

Proposition 5.12. *Let τ be a local section of N^* on $[0, 1]$.*

- (1) $c_E(J(\overline{01}), \tau) \in N_{dR}^*$
- (2) *We set $c_E(J(\overline{01}, \tau)) = v^*$. Let $u_0 \frac{dx}{x} \in E(M, N)_{dR}$. Then we have*

$$(5.4) \quad c_E(J(\overline{10}, \tau))(u_0 \frac{dx}{x}) = \sum_{J_1, J_2} c_{\Phi, J_1 0 J_2} ((P_{J_1} \otimes 1 \otimes 1) Ev(Q_{J_2}(v^*))(u_0),$$

Proof. (1) We have

$$\begin{aligned} J(\overline{01}, \tau) &= \tau_{(\overline{01}, \overline{01})} - \delta(\tau)_{(\overline{01}, \overline{01})} \\ &\in C_0(\overline{01} \times \overline{01}, N_{\overline{01}}^*) \oplus C_0(\overline{01} \times \overline{01}, M_{\overline{01}}^* \otimes M_{\overline{01}} \otimes N_{\overline{01}}^*). \end{aligned}$$

Therefore for $u_0, u_1 \in M_{dR} \otimes M_{dR}^* \otimes N_{dR}$ and $v \in N_{dR}$,

$$c_E(J(\overline{01}, \tau))(u_0 \frac{dx}{x} + u_1 \frac{dx}{x-1} + v) = c_N(\tau)(v).$$

Therefore $c_E(J(\overline{01}, \tau)) \in N_{dR}^*$

(2)

$$\begin{aligned} c_E(J(\overline{10}, \tau))(u_0 \frac{dx}{x}) &= c_E([01]) c_E(J(\overline{01}))(u_0 \frac{dx}{x}) \\ &= c_E([01])(v^*)(u_0 \frac{dx}{x}) \\ &= \sum_{J_1, J_2} c_{\Phi, J_1 0 J_2} ((P_{J_1} \otimes 1 \otimes 1) Ev(Q_{J_2}(v^*))(u_0) \end{aligned}$$

□

6. GENERATING FUNCTION AND ZAGIER'S EXPRESSION

In this section, we show that the formal power series $\Phi(a, b)$ defined in the last section coincides with the formal power series defined in Zagier's paper.

6.1. Junction for hypergeometric modules. Let $V = \mathbf{Q}((a, b))b_1 \oplus \mathbf{Q}((a, b))b_1$ be a free $\mathbf{Q}((a, b))$ module generated by two basis

$$b_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The $\mathbf{Q}((a, b))$ -dual of V is denoted by V^* and the dual basis are denoted by

$$b_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, b_2^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We apply Proposition 5.12 by setting

$$(6.1) \quad \begin{aligned} M &= HM(a_1, b_1, c_1) \otimes x^{-u}, \quad N = HM(a_2, b_2, c_2), \\ P_0 &= \begin{pmatrix} -u & a_1 \\ 0 & -u - c_1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ -b_1 & c_1 - a_1 - b_1 \end{pmatrix}, \\ Q_0 &= \begin{pmatrix} 0 & a_2 \\ 0 & -c_2 \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & 0 \\ -b_2 & c_2 - a_2 - b_2 \end{pmatrix}, \end{aligned}$$

Then the element Δ in Definition 5.11 is equal to

$$\begin{aligned} \Delta &= \frac{a_1 \mathbf{s}(-a_1) \mathbf{s}(b_1 - c_1)}{\mathbf{s}(a_1 + b_1 - c_1)} \gamma_1 y^{-u} \otimes \gamma_1^* x^u + \frac{a_1 \mathbf{s}(a_1 - c_1) \mathbf{s}(b_1)}{\mathbf{s}(c_1 - a_1 - b_1)} \gamma_2 y^{-u} \otimes \gamma_2^* x^u \\ &\in M_B^* \otimes M_B. \end{aligned}$$

by the relation (2.9). Here γ_i, γ_i^* are the topological cycles corresponding to the base §2.3 and §2.3.1. We choose a local section γ of N_B^* on $[0, 1]$ such that the fiber of $\gamma(\overline{01})$ at $\overline{01}$ goes to $c_{N^*}(\gamma(\overline{01})) = b_1^* \in N_{dR}^*$ via the comparison map c_N . We apply Proposition 5.12 by setting $u_0 = b_1 \otimes b_1^* \otimes b_1$

6.1.1. Using Hochschild-Serre-Fubini theorem. Using Hochschild-Serre-Funibi theorem, we have the following lemma.

Proposition 6.1.

$$\begin{aligned} c_E(J(\overline{10}, \gamma))((b_1 \otimes b_1^* \otimes b_1) \frac{dx}{x}) &= \int_{(0,1)} F_\Phi(-a_1, -b_1; c_1 - a_1 - b_1 + 1; 1 - x) x^u \\ &\quad F_\Phi(a_2, b_2; c_2 + 1; x) \frac{dx}{x}. \end{aligned}$$

Proof. We compute the pairing $I_i = \langle \tilde{\gamma}_i, b_1 \otimes b_1^* \otimes b_1 \frac{dx}{x} \rangle$ for $i = 1, 2$ at the fiber at $y = \overline{01}$.

$$\tilde{\gamma} = \{(t_1, t_2, t_3, x) \mid t_1, t_2, t_3 \in [0, 1], x \in (0, 1)\}$$

Then we have

$$\begin{aligned} I_1(y) &= \frac{a_1 \mathbf{s}(-a_1) \mathbf{s}(b_1 - c_1) \Gamma(c_2 + 1)}{\mathbf{s}(a_1 + b_1 - c_1) \Gamma(a_2, c_2 - a_2 + 1)} \\ &\quad \int_{\tilde{\gamma}} t_0^{a_1-1} (1-t_0)^{b_1-c_1-1} (1-(1-y)t_0)^{-b_1} y^{-u} \\ &\quad t_1^{-a_1-1} (1-t_1)^{c_1-b_1} (1-(1-x)t_1)^{b_1} x^u \\ &\quad t_2^{a_2-1} (1-t_2)^{c_2-a_2} (1-xt_2)^{-b_2} \frac{1}{x} dt_0 dt_1 dt_2 dx \end{aligned}$$

$$\begin{aligned}
I_2(y) = (\text{const.}) \int_{\tilde{\gamma}} & (1-y)^{c_1-b_1-a_1+1} t_0^{c_1-a_1-1} (1-t_0)^{-b_1} (1-(1-y)t_0)^{b_1-c_1-1} y^{-u} \\
& (1-x)^{-c_1+a_1+b_1} t_1^{a_1-c_1-1} (1-t_1)^{b_1} (1-(1-x)t_1)^{-b_1+c_1} x^u \\
& t_2^{a_2-1} (1-t_2)^{c_2-a_2} (1-xt_2)^{-b_2} \frac{1}{x} dt_0 dt_1 dt_2 dx
\end{aligned}$$

Therefore by integrating t_0 first, we have $I_2(\overline{10}) = 0$ and

$$\begin{aligned}
I_1(\overline{01}) = & \frac{\Gamma(c_1 - a_1 - b_1 + 1, c_2 + 1)}{\Gamma(-a_1, c_1 - b_1 + 1, a_2, c_2 - a_2 + 1)} \\
& \int_D t_1^{-a_1-1} (1-t_1)^{c_1-b_1} (1-(1-x)t_1)^{b_1} x^u \\
& t_2^{a_2-1} (1-t_2)^{c_2-a_2} (1-xt_2)^{-b_2} \frac{1}{x} dt_1 dt_2 dx,
\end{aligned}$$

where $D = \{(t_1, t_2, x) \mid t_1, t_2 \in [0, 1], x \in (0, 1)\}$. □

6.2. A classical integral formula. To compute the integral of Proposition 6.1, we need an integral formula for associators in Theorem 6.3. Before proving the integral formula (6.4) for associators, we recall a proof of the corresponding classical integral formula.

Lemma 6.2. (1)

$$\begin{aligned}
& \int_{[0,1]^2} s_1^{b_2-a_1-1} (1-s_1)^{a_1-1} {}_2F_1(a_2, a_3; a_1; 1-s_1) ds_1 \\
& = \frac{\Gamma(a_1)\Gamma(b_2-a_1)\Gamma(b_2-a_2-a_3)}{\Gamma(b_2-a_2)\Gamma(b_2-a_3)}.
\end{aligned}$$

(2)

$$\begin{aligned}
& \int_{[0,1]} {}_2F_1(p_1, p_2; q_1, us) s^{b_2-a_1-1} (1-s)^{a_1-1} {}_2F_1(a_2, a_3; a_1; 1-s) ds \\
& = \frac{\Gamma(a_1)\Gamma(b_2-a_1)\Gamma(b_2-a_2-a_3)}{\Gamma(b_2-a_2)\Gamma(b_2-a_3)} \\
& {}_4F_3(p_1, p_2, b_2-a_1, b_2-a_2-a_3; q_1, b_2-a_2, b_2-a_3; u)
\end{aligned}$$

Proof. (1)

$$\begin{aligned}
& \frac{\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_2-a_1)} \int_{[0,1]^2} s_1^{a_1-1} (1-s_1)^{b_2-a_1-1} {}_2F_1(a_2, a_3; a_1; s_1) ds_1 \\
&= \frac{\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_2-a_1)} \frac{\Gamma(a_1)}{\Gamma(a_2)\Gamma(a_1-a_2)} \\
&\quad \int_{[0,1]^2} s_1^{a_1-1} (1-s_1)^{b_2-a_1-1} s_2^{a_2-1} (1-s_2)^{a_1-a_2-1} (1-s_1s_2)^{-a_3} ds_1 ds_2 \\
&= {}_3F_2(a_1, a_2, a_3; b_2, a_1; 1) = {}_3F_2(a_1, a_2, a_3; a_1, b_2; 1) \\
&= {}_2F_1(a_2, a_3; b_2; 1) = \frac{\Gamma(b_2)\Gamma(b_2-a_2-a_3)}{\Gamma(b_2-a_2)\Gamma(b_2-a_3)}
\end{aligned}$$

(2) By the integral expression of hypergeometric function, we have

$$\begin{aligned}
& \int_{[0,1]} {}_2F_1(p_1, p_2; q_1, us) s^{b_2-a_1-1} (1-s)^{a_1-1} {}_2F_1(a_2, a_3; a_1; 1-s) ds \\
&= \sum_m \frac{(p_1)_m (p_2)_m}{(q_1)_m} u^m s^{m+b_2-a_1-1} (1-s)^{a_1-1} {}_2F_1(a_2, a_3; a_1; 1-s) ds.
\end{aligned}$$

By (1), the above is equal to

$$\begin{aligned}
& \sum_m \frac{(p_1)_m (p_2)_m}{(q_1)_m} u^m \frac{\Gamma(a_1)\Gamma(m+b_2-a_1)\Gamma(m+b_2-a_2-a_3)}{\Gamma(m+b_2-a_2)\Gamma(m+b_2-a_3)} \\
&= \frac{\Gamma(a_1)\Gamma(b_2-a_1)\Gamma(b_2-a_2-a_3)}{\Gamma(b_2-a_2)\Gamma(b_2-a_3)} \sum_m \frac{(p_1)_m (p_2)_m (b_2-a_1)_m (b_2-a_2-a_3)_m}{(q_1)_m (b_2-a_2)_m (b_2-a_3)_m} u^m \\
&= \frac{\Gamma(a_1)\Gamma(b_2-a_1)\Gamma(b_2-a_2-a_3)}{\Gamma(b_2-a_2)\Gamma(b_2-a_3)} \\
& \quad {}_4F_3(p_1, p_2, b_2-a_1, b_2-a_2-a_3; q_1, b_2-a_2, b_2-a_3; u).
\end{aligned}$$

□

6.3. Zagier's generating function for associators.

6.3.1. *An integral formula for associators.* Let u_1, u_2, u_3 be the distinguished coordinate of \mathcal{M}_6 . We define admissible functions $s_1, s_2, s_3, x_1, x_2, x_3$ by

$$\begin{aligned}
& (6.2) \\
x_1 &= \frac{(u_1-1)(u_2-u_3)}{(u_1-u_3)(u_2-1)}, \quad x_2 = \frac{(u_2-0)(u_1-u_3)}{(u_2-u_3)(u_1-0)}, \quad x_3 = \frac{(u_1-u_3)(\infty-u_2)}{(u_1-u_2)(\infty-u_3)}, \\
s_1 &= \frac{(u_1-1)(\infty-0)}{(u_1-0)(\infty-1)}, \quad s_2 = \frac{(u_2-0)(\infty-1)}{(u_2-1)(\infty-0)}, \quad s_3 = \frac{(0-u_3)(\infty-1)}{(0-1)(\infty-u_3)}.
\end{aligned}$$

Then we have

$$1 - s_1 = \frac{1}{u_1} = \frac{x_3^*(1-x_2^*x_3^*)}{1-x_1^*x_3}, \quad 1 - x_1x_2 = 1 - s_1s_2 = \frac{(u_2-u_1)(0-1)}{(u_2-1)(0-u_1)}$$

and

$$\frac{x_1}{x_1^* x_3^*} = \frac{s_1}{s_1^* s_3^*}, \quad \frac{x_2}{x_2^* x_3} = \frac{s_2}{s_2^* s_3}, \quad x_1^* x_3 = s_2^* s_3^*, \quad x_2^* x_3^* = s_1^* s_3.$$

As a consequence, we have

$$\begin{aligned} (6.3) \quad & w_1^{p_1} (1 - w_1)^{q_1 - p_1} (1 - \frac{w_1}{u_1})^{-p_2} s_1^{a_1} (1 - s_1)^{b_2 - a_1} s_2^{a_2} (1 - s_2)^{b_1 - a_2} \\ & s_3^{b_2 - a_2} (1 - s_3)^{b_1 - a_1} (1 - s_1 s_2)^{-a_3} \\ & = w_1^{p_1} (1 - w_1)^{q_1 - p_1} (1 - \frac{w_1}{u_1})^{-p_2} (1 - x_1)^{b_1 - a_1} x_1^{a_1} (1 - x_2)^{b_2 - a_2} x_2^{a_2} \\ & (1 - x_3)^{b_2 - a_1} x_3^{b_1 - a_2} (1 - x_1 x_2)^{-a_3} \end{aligned}$$

and

$$\frac{dx_1}{x_1} \frac{dx_2}{x_2} dx_3 = \frac{ds_1}{s_1} \frac{ds_2}{s_2} ds_3$$

The main theorem in this section is the following:

Theorem 6.3. (1)

$$\begin{aligned} (6.4) \quad & \int_{[0,1]}^{\Phi} F_{\Phi}(p_1, p_2; q_1, s) s^{b_2 - a_1 - 1} (1 - s)^{a_1 - 1} F_{\Phi}(a_2, a_3; a_1; 1 - s) ds \\ & = \frac{\Gamma_{\Phi}(a_1, b_2 - a_1, b_2 - a_2 - a_3)}{\Gamma_{\Phi}(b_2 - a_2, b_2 - a_3)} \\ & F_{\Phi}(p_1, p_2, b_2 - a_1, b_2 - a_2 - a_3; q_1, b_2 - a_2, b_2 - a_3; 1) \end{aligned}$$

(2)

$$\begin{aligned} & \int_{(0,1)}^{\Phi} F_{\Phi}(a_2, a_3; 1; 1 - s) s^{b_2 - 1} F_{\Phi}(p_1, p_2; q_1 + 1, s) ds \\ & = \frac{\Gamma_{\Phi}(b_2, b_2 + 1 - a_2 - a_3)}{\Gamma_{\Phi}(b_2 + 1 - a_2, b_2 + 1 - a_3)} \\ & \times F_{\Phi}(p_1, p_2, b_2, b_2 + 1 - a_2 - a_3; q_1 + 1, b_2 + 1 - a_2, b_2 + 1 - a_3; 1) \\ & - \frac{\Gamma_{\Phi}(1 - a_2 - a_3)}{b_1 \Gamma_{\Phi}(1 - a_2, 1 - a_3)} \end{aligned}$$

Proof. Let w_1, u_1, u_2, u_3 be the distinguished coordinate of \mathcal{M}_7 . We consider admissible functions $s_1, s_2, s_3, x_1, x_2, x_3$ on u_1, u_2, u_3 define in (6.2). Using

the relation, (6.3), we have the following equatilty:

(6.5)

$$\int_{[0,1]}^{\Phi} w_1^{p_1} (1-w_1)^{q_1-p_1-1} \left(1 - \frac{w_1}{u_1}\right)^{-p_2} s_1^{a_1} (1-s_1)^{b_2-a_1-1} \\ s_2^{a_2} (1-s_2)^{b_1-a_2-1} s_3^{b_2-a_2-1} (1-s_3)^{b_1-a_1-1} (1-s_1s_2)^{-a_3} ds_3 \frac{ds_1 ds_2 dw_1}{s_1 s_2 w_1}$$

(6.6)

$$= \int_{[0,1]}^{\Phi} w_1^{p_1} (1-w_1)^{q_1-p_1-1} \left(1 - \frac{w_1}{u_1}\right)^{-p_2} (1-x_1)^{b_1-a_1-1} x_1^{a_1} \\ (1-x_2)^{b_2-a_2-1} x_2^{a_2} (1-x_3)^{b_2-a_2-1} x_3^{b_1-a_2-1} (1-x_1x_2)^{-a_3} \frac{dx_1 dx_2 dx_3 dw_1}{x_1 x_2 w_1}$$

We multiply $(b_1 - a_1 + 1)$ with (6.5) and (6.6) and take a limit where b_1 tends to $a_1 - 1$. Using $\lim_{b_1 \rightarrow a_1} (b_1 - a_1) B_\Phi(b_2 - a_2, b_1 - a_1) = 1$, and $\frac{1}{u_1} = 1 - s_1$, the limit of (6.5) is equal to

(6.7)

$$\lim_{b_1 \rightarrow a_1} (b_1 - a_1) \int_{[0,1]}^{\Phi} w_1^{p_1} (1-w_1)^{q_1-p_1-1} \left(1 - \frac{w_1}{u_1}\right)^{-p_2} \\ s_1^{a_1} (1-s_1)^{b_2-a_1-1} s_2^{a_2} (1-s_2)^{b_1-a_2-1} s_3^{b_2-a_2-1} (1-s_3)^{b_1-a_1-1} (1-s_1s_2)^{-a_3} ds_3 \frac{ds_1 ds_2 dw_1}{s_1 s_2 w_1} \\ = \int_{[0,1]}^{\Phi} w_1^{p_1} (1-w_1)^{q_1-p_1-1} (1-w_1(1-s_1))^{-p_2} \\ s_1^{a_1} (1-s_1)^{b_2-a_1-1} s_2^{a_2} (1-s_2)^{a_1-a_2-1} (1-s_1s_2)^{-a_3} \frac{ds_1 ds_2 dw_1}{s_1 s_2 w_1} \\ = \frac{\Gamma_\Phi(p_1, q_1 - p_1, a_2, a_1 - a_2)}{\Gamma_\Phi(q_1, a_1)} \\ \int_{[0,1]}^{\Phi} F_\Phi(p_1, p_2; q_1, 1-s_1) s_1^{a_1-1} (1-s_1)^{b_2-a_1-1} F_\Phi(a_2, a_3; a_1; s_1) ds_1 \\ = \frac{\Gamma_\Phi(p_1, q_1 - p_1, a_2, a_1 - a_2)}{\Gamma_\Phi(q_1, a_1)} \\ \int_{[0,1]}^{\Phi} F_\Phi(p_1, p_2; q_1, s) s^{b_2-a_1-1} (1-s)^{a_1-1} F_\Phi(a_2, a_3; a_1; 1-s) ds$$

We compute the limit of (6.6) using Proposition 4.11. Since $\lim_{x_1 \rightarrow 1} \frac{1}{u_1} = x_2^* x_3^*$, we have

(6.8)

$$\begin{aligned}
& \lim_{b_1 \rightarrow a_1} (b_1 - a_1) \int_{[0,1]}^\Phi w_1^{p_1} (1 - w_1)^{q_1 - p_1 - 1} \left(1 - \frac{w_1}{u_1}\right)^{-p_2} (1 - x_1)^{b_1 - a_1 - 1} x_1^{a_1} \\
& \quad (1 - x_2)^{b_2 - a_2 - 1} x_2^{a_2} (1 - x_3)^{b_3 - a_3 - 1} x_3^{b_3 - a_3 - 1} (1 - x_1 x_2)^{-a_3} \frac{dx_1 dx_2 dx_3 dw_1}{x_1 x_2 w_1} \\
&= \int_{[0,1]}^\Phi w_1^{p_1} (1 - w_1)^{q_1 - p_1 - 1} (1 - w_1 x_2^* x_3^*)^{-p_2} \\
& \quad (1 - x_2)^{b_2 - a_2 - a_3 - 1} x_2^{a_2} (1 - x_3)^{b_3 - a_3 - 1} x_3^{a_3 - a_2} \frac{dx_2 dx_3 dw_1}{x_2 x_3 w_1} \\
&= \int_{[0,1]}^\Phi w_1^{p_1} (1 - w_1)^{q_1 - p_1 - 1} (1 - w_1 x_2 x_3)^{-p_2} \\
& \quad x_2^{b_2 - a_2 - a_3} (1 - x_2)^{a_2 - 1} x_3^{b_3 - a_3} (1 - x_3)^{a_3 - a_2 - 1} \frac{dx_2 dx_3 dw_1}{x_2 x_3 w_1} \\
&= \frac{\Gamma_\Phi(p_1, q_1 - p_1, b_2 - a_2 - a_3, a_2, b_2 - a_1, a_1 - a_2)}{\Gamma_\Phi(q_1, b_2 - a_3, b_2 - a_2)} \\
& \quad F_\Phi(p_1, b_2 - a_2 - a_3, b_2 - a_1, p_2; q_1, b_2 - a_3, b_2 - a_2; 1)
\end{aligned}$$

By comparing two limits (6.7) and (6.8), we have the theorem.

(2) By replacing q_1 and a_1 by $q_1 + 1$ and 1, we have

$$\begin{aligned}
& \int_{[0,1]}^\Phi F_\Phi(a_2, a_3; 1; 1 - s) s^{b_2 - 1} F_\Phi(p_1, p_2; q_1 + 1, s) ds \\
&= \frac{\Gamma_\Phi(b_2, b_2 + 1 - a_2 - a_3)}{\Gamma_\Phi(b_2 + 1 - a_2, b_2 + 1 - a_3)} \\
& \quad F_\Phi(p_1, p_2, b_2, b_2 + 1 - a_2 - a_3; q_1 + 1, b_2 + 1 - a_2, b_2 + 1 - a_3; 1)
\end{aligned}$$

By Proposition 4.14, we have the statement (2). \square

As a corollary, we have the following corollary.

Corollary 6.4. *Let P, Q be matrices in (6.1) evaluated at $c_1 = 0$. Then we have*

$$\begin{aligned}
& \sum_{J_1, J_2, \deg(J_2) > 0} c_{\Phi, J_1 0 J_2}(b_1, P_{J_1} b_1^* b_1 Q_{J_2} b_1^*) \\
&= \frac{\Gamma_\Phi(u, u + 1 - a_1 - b_1)}{\Gamma_\Phi(u + 1 - a_1, u + 1 - b_1)} \\
& \quad \left[F_\Phi(a_2, b_2, u, u + 1 - a_1 - b_1; c_2 + 1, u + 1 - a_1, u + 1 - b_1; 1) - 1 \right]
\end{aligned}$$

By setting $a_2 = a, b_2 = -a, a_1 = -b, b_1 = b$ and taking the limit for $u \rightarrow 0$, we have the following theorem.

Theorem 6.5. *We have the following equality:*

$$\sum_{n \geq 0, m > 0} c_{\Phi, (01)^n 0 (01)^m} b^{2n+1} a^{2m} = \mathbf{s}(b) \frac{d}{dz} |_{z=0} F_{\Phi}(a, -a, z; 1+b, 1-b; 1).$$

7. SELBERG INTEGRAL AND DIXON'S THEOREM

7.1. Φ -Selberg integral formula. Let x_1, x_2 be the distinguished coordinate of \mathcal{M}_5 . We define even and odd Φ -Selberg integrals by

$$S_{\Phi}^+(a, b, c) = \int_{0 \leq x_2 \leq x_1 \leq 1}^{\Phi} (x_1 x_2)^{a-1} ((1-x_1)(1-x_2))^{b-1} (x_1 - x_2)^{2c} dx_1 dx_2$$

$$S_{\Phi}^-(a, b, c) = \int_{0 \leq x_2 \leq x_1 \leq 1}^{\Phi} (x_1 x_2)^{a-1} ((1-x_1)(1-x_2))^{b-1} (x_1 - x_2)^{2c+1} dx_1 dx_2$$

7.1.1. Variant. Following the idea of Aomoto (1984) and Lavoie-Grondin-Rathie-Arora (1994), we consider the following variant. For a polynomial $f(x, y)$ of x, y , we set

$$Sel_{\Phi}(f(x, y))_{a,b,c} = \int_{0 \leq x \leq y \leq 1}^{\Phi} f(x, y) x^{a-1} y^{a-1} (1-x)^{b-1} (1-y)^{b-1} (y-x)^{2c} dx dy$$

The following lemma is direct from the definition.

Lemma 7.1. (1)

$$Sel_{\Phi}(1)_{a,b,c} = S_{\Phi}^+(a, b, c),$$

$$Sel_{\Phi}(y-x)_{a,b,c} = S_{\Phi}^-(a, b, c),$$

(2)

$$Sel_{\Phi}(xyf(x, y))_{a,b,c} = Sel_{\Phi}(f(x, y))_{a+1,b,c}$$

$$Sel_{\Phi}((1-x)(1-y)f(x, y))_{a,b,c} = Sel_{\Phi}(f(x, y))_{a,b+1,c}$$

Using the equalities in the lemma, for any polynomial $f(x, y)$, $Sel_{\Phi}(f(x, y))_{a,b,c}$ can be computed using Selberg integrals $S_{\Phi}^+(a, b, c)$ and $S_{\Phi}^-(a, b, c)$.

7.2. Even Selberg integral. In this subsection, we prove the following proposition.

Proposition 7.2 (Φ^+ -Selberg integral formula).

$$S_{\Phi}^+(a, b, c) = \frac{\Gamma_{\Phi}(a, b, a+c, b+c, 2c)}{\Gamma_{\Phi}(c, a+b+c, a+b+2c)}$$

We define \mathcal{A}_6^{Φ} module

$$T(a, b, c) = x_1^a x_3^a (1-x_1)^b (1-x_3)^b (x_2 - x_1)^c (x_3 - x_2)^c \mathbf{Q}[[a, b, c]]$$

Let $\iota : \mathcal{M}_6 \rightarrow \mathcal{M}_6$ be an involution defined by $(x_1, x_3) \mapsto (x_3, x_1)$. Then we have an equivariant action $T(a, b, c) \simeq \iota_* T(a, b, c)$. The real valued section at

$p^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ ($0 < x_1^{(0)} < x_2^{(0)} < x_3^{(0)} < 1$) goes to the section with real value in

$$\begin{aligned}\iota^* T(a, b, c)_{p^{(0)}} &= T(a, b, c)_{\iota(p^{(0)})} \\ &= (x_3^{(0)})^a (x_1^{(0)})^a (1 - x_3^{(0)})^b (1 - x_1^{(0)})^b (x_2^{(0)} - x_3^{(0)})^c (x_1^{(0)} - x_2^{(0)})^c \mathbf{Q}[[a, b, c]]\end{aligned}$$

We consider the pairing

$$H_3^\Phi(\mathcal{M}_6, T(a, b, c)) \otimes H_\Phi^3(\mathcal{M}_6, T(-a, -b, -c)) \rightarrow \mathbf{C}[[a, b, c]].$$

7.2.1. *The first integral formula.* Let $\mathcal{M}_6 \rightarrow \mathcal{M}_5$ be the map defined by $(x_1, x_2, x_3) \mapsto (x_1, x_3)$.

$$\begin{aligned}& \int_{0 < x_1 < x_3 < 1}^\Phi \left[\int_{x_1 < x_2 < x_3}^\Phi x_1^a x_3^a (1 - x_1)^b (1 - x_3)^b \right. \\ & \quad \left. (x_2 - x_1)^c (x_3 - x_2)^c (x_3 - x_1) dx_2 \right] dx_1 dx_3 \\ &= \int_{0 < x_1 < x_3 < 1}^\Phi \left[x_1^a x_3^a (1 - x_1)^b (1 - x_3)^b (x_3 - x_1) \right. \\ & \quad \left. \int_{x_1 < x_2 < x_3}^\Phi (x_3 - x_2)^c (x_2 - x_1)^c dx_2 \right] dx_1 dx_3 \\ &= \frac{\Gamma_\Phi(c+1)^2}{\Gamma_\Phi(2c+2)} \int_{0 < x_1 < x_3 < 1}^\Phi \left[x_1^a x_3^a (1 - x_1)^b (1 - x_3)^b (x_3 - x_1)^{2c+2} \right] dx_1 dx_3 \\ &= \frac{\Gamma_\Phi(c+1)^2}{\Gamma_\Phi(2c+2)} S_\Phi(a+1, b+1, c+1)\end{aligned}$$

7.2.2. *The second integral formula.* We need the following formula. Let $q : \mathcal{M}_6 \rightarrow \mathcal{M}_4$ be the map defined by $(x_1, x_2, x_3) \mapsto x_2$. We define $\mathcal{D}(a, b, c)$ by the fixed part of $\mathbf{R}^2 q_*(T(a, b, c))^\iota$.

Proposition 7.3 (Determinant formula). (1) $\mathcal{D}(a, b, c)_{dR}$ is a torsion free sheaf of rank one over $\mathbf{Q}[[a, b, c]]$ generated by

$$x_1^a x_3^a (1 - x_1)^b (1 - x_3)^b (x_2 - x_1)^c (x_3 - x_2)^c (x_3 - x_1) dx_1 dx_3.$$

(2) We have an isomorphism of \mathcal{A}_4^Φ -modules

$$\phi : \mathcal{D}(a, b, c) \simeq \mathbf{B}_\Phi(a, c) \otimes \mathbf{B}_\Phi(a+c, b) \otimes_{\mathbf{Q}[[a, b, c]]} \left(t^{a+c} (1-t)^{b+c} \mathbf{Q}[[a, b, c]] \right)$$

Proof. (2) We consider the \mathcal{A}_4^Φ -module M defined by

$$M = \mathcal{D}(a, b, c) \otimes_{\mathbf{Q}[[a, b, c]]} \left(t^{-a-c} (1-t)^{-b-c} \mathbf{Q}[[a, b, c]] \right)$$

We compute the action of A_{dR}^4 on M_{dR} . Then the map

$$M_{dR} \xrightarrow{e_0, e_1} M_{dR} \oplus M_{dR}$$

is the zero map. Therefore M is the pull back of an object in \mathcal{C} . To consider the fiber at $\overline{01}$, we consider the integral

$$\begin{aligned} & x_2^{-a-c-1} \int_{0 \leq x_1 \leq t, t \leq x_3 \leq 1}^{\Phi} x_1^a x_3^a (1-x_1)^b (1-x_3)^b (x_2-x_1)^c (x_3-x_2)^c (x_3-x_1) dx_1 dx_3 \\ &= \int_{0 \leq \xi \leq 1, x_2 \leq x_3 \leq 1}^{\Phi} \xi^a x_3^a (1-x_2 \xi)^b (1-x_3)^b (1-\xi)^c (x_3-x_2)^c (x_3-x_2 \xi) d\xi dx_3 \\ &\xrightarrow{x_2 \rightarrow 0} \int_{0 \leq \xi \leq 1, 0 \leq x_3 \leq 1}^{\Phi} \xi^a x_3^{a+c+1} (1-x_3)^b (1-\xi)^c d\xi dx_3 \end{aligned}$$

Therefore the fiber $M_{\overline{01}} \in \mathcal{C}$ is isomorphic to $\mathbf{B}_{\Phi}(a, c) \otimes \mathbf{B}_{\Phi}(a+c, b)$. Thus we have the proposition. \square

Using the above proposition, we have

$$\begin{aligned} & \int_{0 < x_2 < 1}^{\Phi} \left[\int_{0 < x_1 < x_2, x_2 < x_3 < 1}^{\Phi} x_1^a x_3^a (1-x_1)^b (1-x_3)^b \right. \\ & \quad \left. (x_2-x_1)^c (x_3-x_2)^c (x_3-x_1) dx_1 dx_3 \right] dx_2 \\ &= \frac{\Gamma_{\Phi}(a+1, c+1, a+c+2, b+1)}{\Gamma_{\Phi}(a+c+2, a+b+c+3)} \int_{0 < x_2 < 1}^{\Phi} x_2^{a+c+1} (1-x_2)^{b+c+1} dx_2 \\ &= \frac{\Gamma_{\Phi}(a+1, c+1, b+1, a+c+2, b+c+2)}{\Gamma_{\Phi}(a+b+c+3, a+b+2c+4)} \end{aligned}$$

7.2.3. Proof of Φ -Selberg integral formula. By computing the integral

$$\int_{0 < x_1 < x_2 < x_3 < 1}^{\Phi} x_1^a x_3^a (1-x_1)^b (1-x_3)^b (x_2-x_1)^c (x_3-x_2)^c (x_3-x_1) dx_1 dx_2 dx_3$$

in two ways, we have

$$\begin{aligned} & \frac{\Gamma(a+1)\Gamma(c+1)\Gamma(b+1)\Gamma(a+c+2)\Gamma(b+c+2)}{\Gamma(a+b+c+3)\Gamma(a+b+2c+4)} \\ &= \frac{\Gamma_{\Phi}(c+1)^2}{\Gamma_{\Phi}(2c+2)} S_{\Phi}^+(a+1, b+1, c+1) \end{aligned}$$

and

$$\frac{\Gamma_{\Phi}(2c+2, a+1, b+1, a+c+2, b+c+2)}{\Gamma_{\Phi}(c+1, a+b+c+3, a+b+2c+4)} = S_{\Phi}^+(a+1, b+1, c+1)$$

7.3. Odd Φ -Selberg integral. We compute the following odd Φ -Selberg integral $S_{\Phi}^-(a, b, c)$

First we consider the differential equations satisfied by ${}_3F_2$. For a cycle γ , we consider the integral

$$f(\gamma) = \int_{\gamma}^{\Phi} t_1^{a_1} (1-t_1)^{c_1-a_1} t_2^{a_2} (1-t_2)^{c_2-a_2} (1-xt_1t_2)^{-a_3} \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

We change coordinates to (t_2, t_3) with the relation $xt_1t_2t_3 = 1$. It is also expressed by similar integral expression. Using this expression,

$$\begin{aligned} f_{11} &= r_1 F_{\Phi}(a_1, a_2, a_3; c_1 + 1, c_2 + 1; x) \\ f_{12} &= r_2 x^{-c_1} F_{\Phi}(a_2 - c_1, a_3 - c_1, a_1 - c_1; c_2 - c_1 + 1, -c_1 + 1; x) \\ f_{13} &= r_3 x^{-c_2} F_{\Phi}(a_1 - c_2, a_3 - c_2, a_2 - c_2; c_1 - c_2 + 1, -c_2 + 1; x) \end{aligned}$$

with

$$\begin{aligned} r_1 &= B_{\Phi}(a_1, c_1 - a_1 + 1) B_{\Phi}(a_2, c_2 - a_2 + 1) \\ r_2 &= B_{\Phi}(a_2 - c_1, c_2 - a_2 + 1) B_{\Phi}(a_3 - c_1, -a_3 + 1) \\ r_3 &= B_{\Phi}(a_1 - c_2, c_1 - a_1 + 1) B_{\Phi}(a_3 - c_2, -a_3 + 1) \end{aligned}$$

satisfies the same rational differential equation of t . Thus we have

$$\begin{aligned} r_1 r_2 r_3 &= \frac{\Gamma_{\Phi}(a_1, c_1 - a_1 + 1, a_2, c_2 - a_2 + 1)}{\Gamma_{\Phi}(c_1 + 1, c_2 + 1)} \\ &\quad \frac{\Gamma_{\Phi}(a_2 - c_1, c_2 - a_2 + 1, a_3 - c_1, -a_3 + 1)}{\Gamma_{\Phi}(-c_1 + c_2 + 1, -c_1 + 1)} \\ &\quad \frac{\Gamma_{\Phi}(a_1 - c_2, c_1 - a_1 + 1, a_3 - c_2, -a_3 + 1)}{\Gamma_{\Phi}(-c_2 + c_1 + 1, -c_2 + 1)} \end{aligned}$$

They are integral of the following chains up to constant:

$$\gamma_1 = \{(t_1, t_2) \in [0, 1]^2\}, \quad \gamma_2 = \{(t_2, t_3) \in [0, 1]^2\}, \quad \gamma_3 = \{(t_1, t_3) \in [0, 1]^2\}.$$

We define a matrix $F = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = (f_{ij})$ by and

$$f_{2i} = x \frac{df_{1i}}{dx}, \quad f_{3i} = x \frac{df_{2i}}{dx}.$$

In general, we set

$$\mathbf{f}(\gamma) = \begin{pmatrix} f(\gamma) \\ x \frac{d}{dx} f(\gamma) \\ x \frac{d}{dx} x \frac{d}{dx} f(\gamma) \end{pmatrix}, \quad \mathbf{f}'(\gamma) = \lim_{t \rightarrow 1} \left(x \frac{d}{dx} f(\gamma) \right)$$

By setting

$$\begin{aligned} P &= \frac{dx}{x} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -c_2 c_1 & -c_2 - c_1 \end{pmatrix} \\ &\quad + \frac{dx}{x-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_1 a_2 a_3 & -a_2 a_1 + c_2 c_1 - a_3 a_1 - a_3 a_2 & -a_1 - a_2 - a_3 + c_1 + c_2 \end{pmatrix}, \end{aligned}$$

we have

$$dF = PF.$$

Therefore

$$\det(F) = cx^{-c_1-c_2}(1-x)^{-a_1-a_2-a_3+c_1+c_2}$$

with some constant c . By considering the limit for $x = 0$, we have

$$c = r_1 r_2 r_3 c_1 c_2 (c_1 - c_2).$$

Now we consider the limit for $x = 1$. Changing coordinate $\xi_1 = \frac{1}{t_1}$, $\xi_2 = t_2$, we have

$$\begin{aligned} & \int_{\gamma}^{\Phi} t_1^{a_1} (1-t_1)^{c_1-a_1} t_2^{a_2} (1-t_2)^{c_2-a_2} (1-xt_1t_2)^{-a_3} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\ &= \int_{\gamma}^{\Phi} \xi_1^{a_3-c_1} (\xi_1-1)^{c_1-a_1} \xi_2^{a_2} (1-\xi_2)^{c_2-a_2} (\xi_1-x\xi_2)^{-a_3} \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} \end{aligned}$$

Under this coordinate, γ_2 is equal to

$$\gamma_2 = \{0 \leq \xi_1 \leq x\xi_2, 0 \leq \xi_2 \leq 1\}.$$

We set

$$\begin{aligned} \gamma_5 &= \{x\xi_2 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1\} \\ \tau &= \{x\xi_2 \leq \xi_1 \leq 1, 1 \leq \xi_2 \leq \frac{1}{x}\} \end{aligned}$$

Using the equality

$$\begin{aligned} & \sin(\pi\alpha) \int_0^u x^\alpha (u-x)^\beta (1-x)^\gamma \frac{dx}{x} + \sin(\pi(\alpha+\beta)) \int_u^1 x^\alpha (x-u)^\beta (1-x)^\gamma \frac{dx}{x} \\ &+ \sin(\pi(\alpha+\beta+\gamma)) \int_1^\infty x^\alpha (x-u)^\beta (x-1)^\gamma \frac{dx}{x} = 0 \end{aligned}$$

we have the following relations for topological cycles.

Lemma 7.4. *We have*

$$\begin{aligned} \gamma_5 &= -\frac{\sin(\pi a_1)}{\sin(\pi c_1)} \gamma_1 + \frac{\sin(\pi(a_3 - c_1))}{\sin(\pi c_1)} \gamma_2 \\ \tau &= k_1 \gamma_1 + k_2 \gamma_2 + k_3 \gamma_3 \end{aligned}$$

with

$$k_3 = \frac{\sin(\pi(c_2 - a_1)) \sin(\pi(c_2 - a_3))}{\sin(\pi c_2) \sin(\pi(c_1 - c_2))}.$$

As a consequence, we have

$$\det(\mathbf{f}(\gamma_2), \mathbf{f}(\gamma_5), \mathbf{f}(\tau)) = C x^{-c_1-c_2} (1-x)^{c_1+c_2-a_1-a_2-a_3}$$

with

$$\begin{aligned} C &= \Gamma_{\Phi}(a_2, c_1 - a_1 + 1, c_2 - a_2 + 1, c_2 - a_2 + 1, -a_3 + 1) \\ &\quad \frac{\Gamma_{\Phi}(a_2 - c_1, c_1 - a_1 + 1, a_3 - c_1, -a_3 + 1)}{\Gamma_{\Phi}(-a_1 + 1, c_2 - a_1 + 1, c_2 - a_3 + 1)} \end{aligned}$$

By taking a limit $t \rightarrow 1$, since τ is a vanishing cycle, we have the following theorem.

Proposition 7.5. (1)

$$\lim_{t \rightarrow 1} (1-t)^{-c_1-c_2+a_1+a_2+a_3} \mathbf{f}(\tau) = \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix},$$

$$J = \frac{\Gamma_\Phi(-a_3+1, c_1-a_1+1, c_2-a_2+1)}{\Gamma_\Phi(c_1+c_2-a_1-a_2-a_3+1)}$$

(2)

$$\det(\mathbf{f}'(\gamma_2), \mathbf{f}'(\gamma_5))$$

$$= \Gamma_\Phi(a_2, c_1 + c_2 - a_1 - a_2 - a_3 + 1, a_2 - c_1, c_1 - a_1 + 1, a_3 - c_1)$$

$$\frac{\Gamma_\Phi(c_2 - a_2 + 1, -a_3 + 1)}{\Gamma_\Phi(-a_1 + 1, c_2 - a_1 + 1, c_2 - a_3 + 1)}$$

Corollary 7.6.

Proof. We use Proposition 7.5, by settning $-a_3 - 1 = 2c, c_1 - a_1 = c_2 - a_2 = b - 1, a_2 = a_3 - c_1 = a$ hand have

$$(2c+1)S_\Phi^-(a, b, c)Sel_\Phi(x+y)_{a,b,c}$$

$$= \det \begin{pmatrix} S_\Phi^-(a, b, c) & -S_\Phi^-(a, b, c) \\ (2c+1)Sel_\Phi(x)_{a,b,c} & -(2c+1)Sel_\Phi(y)_{a,b,c} \end{pmatrix}$$

$$= \frac{\Gamma_\Phi(a, a, b, b, 2c+2b, 2a+2c+1, 2c+2)}{\Gamma_\Phi(2c+a+b+1, 2c+a+b+1, 2a+2b+2c)}$$

Since

$$Sel_\Phi(x+y)_{a,b,c} = Sel_\Phi(1)_{a+1,b,c} - Sel_\Phi(1)_{a,b+1,c} + Sel_\Phi(1)_{a,b,c}$$

$$= \frac{2\Gamma_\Phi(a+1+c, 2c, b+c, b, a)}{\Gamma_\Phi(c, a+1+b+2c, a+b+c)}$$

We have

$$S_\Phi^-(a, b, c)$$

$$= \frac{\Gamma_\Phi(a, b, c+1, a+b+c, 2c+2b, 2a+2c+1)}{\Gamma_\Phi(2a+2b+2c, b+c, a+c+1, 2c+a+b+1)}$$

□

7.4. Φ -Dixon's theorem and its variants. We prove Dixon's theorem and its generalization using Selberg integral formula.

Proposition 7.7 (Dixson's theorem and its variant). (1)

$$\begin{aligned} & F_{\Phi}(2c, b, a; 2c - a + 1, 2c - b + 1; 1) \\ &= \frac{\Gamma_{\Phi}(1 + c, 1 + 2c - a, 1 + 2c - b, 1 + c - a - b)}{\Gamma_{\Phi}(1 + 2c, 1 + c - a, 1 + c - b, 1 + 2c - a - b)} \end{aligned}$$

(2)

$$\begin{aligned} & F_{\Phi}(2c + 1, b, a; 2c - a + 3, 2c - b + 3; 1) \\ &= \left(\Gamma_{\Phi}(2c, 2c + 4 - 2a - 2b, c + 1 - b, c + 1 - a, c + 2 - a, c + 2 - b) \right. \\ &\quad \left. - \Gamma_{\Phi}(2c + 2 - 2b, 2c + 2 - 2a, c, c + 2 - a - b, c + 3 - a - b, c + 1) \right) \\ &\quad \frac{\Gamma_{\Phi}(2c - b + 3, 2c - a + 3)}{2(b-1)(a-1)\Gamma_{\Phi}(2c+3-a-b, c+2-a, c+2-b, 2c+2-2b, 2c+2-2a, c, c+2-a-b, 2c+1)} \end{aligned}$$

(3)

$$\begin{aligned} & F_{\Phi}(2c + 1, 1, a; 2c - a + 3, 2c + 2; 1) \\ &= - \frac{(2c + 1)(a - 2c - 2)(\Psi(2c + 1) - \Psi(c + 1) - \Psi(2c + 3 - 2a) + \Psi(c + 2 - a))}{(a - 1)} \end{aligned}$$

Here we define

$$\Psi(x) = \frac{d}{dx} \log \Gamma_{\Phi}(z).$$

Proof.

$$\begin{aligned} & \frac{\Gamma_{\Phi}(2c, -a + 1, b, 2c - 2b + 1)}{\Gamma_{\Phi}(2c - a + 1, 2c - b + 1)} F_{\Phi}(2c, b, a; 2c - a + 1, 2c - b + 1; 1) \\ &= \int_{[0,1]^2}^{\Phi} x^{2c-1} (1-x)^{-a} y^{b-1} (1-y)^{2c-2b} (1-xy)^{-a} dx dy \\ &= \int_{0 \leq t \leq x \leq 1}^{\Phi} x^{b-1} (1-x)^{-a} t^{b-1} (x-t)^{2c-2b} (1-t)^{-a} dx dt \\ &= S_{\Phi}^+(b, -a + 1, c - b) \end{aligned}$$

Here we change variables by $(x, t) \rightarrow (x, y) = (x, t/x)$. By Selberg integral formula, we have the proposition.

(2) Using even and odd Selberg integrals, we have

$$\begin{aligned} & \frac{\Gamma_{\Phi}(2c + 1, -a + 2, b, 2c - 2b + 3)}{\Gamma_{\Phi}(2c - a + 3, 2c - b + 3)} F_{\Phi}(2c + 1, b, a; 2c - a + 3, 2c - b + 3; 1) \\ &= \int_{[0,1]^2}^{\Phi} x^{2c} (1-x)^{-a+1} y^{b-1} (1-y)^{2c-2b+2} (1-xy)^{-a} dx dy \\ &= \int_{0 \leq t \leq x \leq 1}^{\Phi} x^{b-2} (1-x)^{-a+1} t^{b-1} (x-t)^{2c-2b+2} (1-t)^{-a} dx dt \\ &= Sel_{\Phi}((1-x)y)_{b-1, -a+1, c-b+1} \end{aligned}$$

Therefore we have the proposition

(3) We take a limit $b \rightarrow 1$.

$$\begin{aligned} & F_\Phi(2c+1, 1, a; 2c-a+3, 2c+2; 1) \\ &= \frac{1}{2} \lim_{b \rightarrow 1} Sel_\Phi(1 - (1-x)(1-y) - (x-y) - xy)_{b-1, -a+1, c-b+1} \\ &= -\frac{(2c+1)(a-2c-2)(\Psi(2c+1) - \Psi(c+1) - \Psi(2c+3-2a) + \Psi(c+2-a))}{(a-1)} \end{aligned}$$

□

8. BROWN-ZAGIER RELATION FOR ASSOCIATORS

8.1. Li's computation for Brown-Zagier relation. We have the following relations between $F_\Phi(a_1, a_1, a_3; b_1, b_2) = F_\Phi(a_1, a_1, a_3; b_1, b_2; 1)$ arising from relations in $H_2^{\Phi, B}(\mathcal{M}_5, \mathcal{F}(a_1, a_2, a_3; b_1, b_2))$ and $H_{\Phi, dR}^2(\mathcal{M}_5, \mathcal{F}(a_1, a_2, a_3; b_1, b_2))$.

Proposition 8.1. *We have the following equalities*

(1)

$$\begin{aligned} & F_\Phi(x, -x, z; 1+y, 1-y) \\ &= \frac{1}{2} F_\Phi(x, 1-x, z; 1+y, 1-y) + \frac{1}{2} F_\Phi(1+x, -x, z; 1+y, 1-y) \end{aligned}$$

(2)

$$\begin{aligned} (8.1) \quad & F_\Phi(x, 1-x, z; 1+y, 1-y) \\ &= \frac{\Gamma_\Phi(1+y, 1-x+y-z)}{\Gamma_\Phi(1-x+y, 1+y-z)} F_\Phi(x, x-y, z; x-y+z, 1-y) \\ &+ \frac{\Gamma_\Phi(1+y, 1-y, x-y+z-1, 1-z)}{\Gamma_\Phi(x, z, x-y, 2-y-z)} \\ & F_\Phi(1-x+y, 1+y-z, 1-z; 2-x+y-z, 2-x-z) \end{aligned}$$

(3)

$$\begin{aligned} & F_\Phi(x, x-y, z; x-y+z, 1-y) \\ &= \frac{\Gamma_\Phi(1-x-y, 1-y)}{\Gamma_\Phi(1-y-z, 1-x-y+z)} F_\Phi(-y+z, z, z; x-y+z, 1-x-y+z) \end{aligned}$$

Proof. The equality (1) follows from an equality in de Rham cohomology. The equality (2) follows from an equality for Betti cohomology and change of coordinates. The equality (3) follows by changing coordinate of integral expression. □

Since

$$F_\Phi(a_1, a_2, a_3; c_1, c_2) \in 1 + a_1 \mathbf{C}[[a_1, \dots, c_2]],$$

we have the following proposition.

Lemma 8.2.

$$\frac{d}{dz} F_\Phi(-y+z, z, z; x-y+z, 1-x-y+z) |_{z=0} = 0$$

Now we are ready to compute the function in Theorem 6.5 using Li's computation.

Proposition 8.3. *We have*

$$\begin{aligned} & \frac{d}{dz} F_\Phi(x, 1-x, z; 1+y, 1-y) \\ &= \Psi(1+y) + \Psi(1-y) - \Psi(1-x+y) - \Psi(1-x-y) \\ &\quad - \frac{\mathbf{s}(x)}{\mathbf{s}(y)} (\Psi(1-x+y) - \Psi(1-x-y) - \Psi(1 - \frac{x-y}{2}) + \Psi(1 - \frac{x+y}{2})) \end{aligned}$$

Proof. We compute the each term of the derivative of (8.1). Using Proposition 8.1 (3) and Lemma 8.2, we have

$$\begin{aligned} & \frac{d}{dz} \left(\frac{\Gamma_\Phi(1+y, 1-x+y-z)}{\Gamma_\Phi(1-x+y, 1+y-z)} F_\Phi(x, x-y, z; x-y+z, 1-y) \right)_{z=0} \\ &= \frac{d}{dz} \left(\frac{\Gamma_\Phi(1+y, 1-y, 1-x-y, 1-x+y-z)}{\Gamma_\Phi(1-x+y, 1+y-z, 1-y-z, 1-x-y+z)} \right. \\ &\quad \left. F_\Phi(-y+z, z, z; x-y+z, 1-x-y+z) \right)_{z=0} \\ &= \Psi(1+y) + \Psi(1-y) - \Psi(1-x+y) - \Psi(1-x-y) \end{aligned}$$

We compute the derivative of the second term of (8.1). Since $\lim_{z \rightarrow 0} \Gamma_\Phi(z)z = 1$, we have

$$\begin{aligned} & \frac{d}{dz} \left(\frac{\Gamma_\Phi(1+y, 1-y, x-y+z-1, 1-z)}{\Gamma_\Phi(x, z, x-y, 2-y-z)} \right. \\ &\quad \left. F_\Phi(1-x+y, 1+y-z, 1-z; 2-x+y-z, 2-x-z) \right)_{z=0} \\ &= \frac{\Gamma_\Phi(1+y, 1-y, x-y-1)}{\Gamma_\Phi(x, x-y, 2-x)} F_\Phi(1-x+y, 1+y, 1; 2-x+y, 2-x) \\ &= \frac{y\mathbf{s}(x)}{(x-1)(x-y-1)\mathbf{s}(y)} F_\Phi(1-x+y, 1+y, 1; 2-x+y, 2-x) \end{aligned}$$

By setting $a = y+1, 2c = y-x$ in the equality of Proposition 8.1 (2), it is equal to

$$\frac{\mathbf{s}(x)}{\mathbf{s}(y)} (\Psi(y-x+1) - \Psi(1 + \frac{y-x}{2}) - \Psi(1-x-y) - \Psi(1 - \frac{x+y}{2}))$$

Thus we have the proposition. \square

Proof of Theorem 1.1. By Theorem 6.5, and Proposition 8.1 (1) and Proposition 8.3, we have

$$\begin{aligned}
& \sum_{n \geq 0, m > 0} c_{\Phi, (01)^n 0 (01)^m} y^{2n+1} x^{2m} \\
&= \mathbf{s}(y) \frac{d}{dz} \Big|_{z=0} F_{\Phi}(x, -x, z; 1+y, 1-y; 1) \\
&= \frac{\mathbf{s}(y)}{2} \frac{d}{dz} \Big|_{z=0} \left(F_{\Phi}(x, 1-x, z; 1+y, 1-y) + F_{\Phi}(1+x, -x, z; 1+y, 1-y) \right) \\
&= \frac{\mathbf{s}(y)}{2} \left(2\Psi(1+y) + 2\Psi(1-y) - \Psi(1+x+y) \right. \\
&\quad \left. - \Psi(1-x-y) - \Psi(1+x-y) - \Psi(1-x+y) \right) \\
&\quad - \frac{\mathbf{s}(x)}{2} \left(\Psi(1+\frac{x+y}{2}) + \Psi(1-\frac{x+y}{2}) - \Psi(1+\frac{x-y}{2}) - \Psi(1+\frac{y-x}{2}) \right. \\
&\quad \left. - \Psi(1+x+y) - \Psi(1-x-y) + \Psi(1+x-y) + \Psi(1-y-x) \right)
\end{aligned}$$

Using the equality

$$\Psi(1+z) = \frac{d}{dx} \log \Gamma_{\Phi}(z+1) = \sum_{n=2}^{\infty} (-1)^n \zeta_{\Phi}(n) x^{n-1}$$

we have the theorem. \square

REFERENCES

- [B] Francis, Brown., Mixed Tate motives over \mathbf{Z} , *Ann of Math.*, **175** (2012), 949–976.
- [Z] Zagier, Don, Evaluation of the multiple zeta value $\zeta(2, \dots, 2, 3, \dots, 2)$ *Ann of Math.*, **175** (2012), 977–1000.
- [L] Li, Zhonghua., Another proof of Zagier’s evaluation formula of the multiple zeta values $\zeta(2, \dots, 2, 3, \dots, 3)$, arXiv:1204.2060v1